

# NEW APPROACHES TO FINITE GENERATION OF COHOMOLOGY RINGS

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ABSTRACT. In support variety theory, representations of a finite dimensional (Hopf) algebra  $A$  can be studied geometrically by associating any representation of  $A$  to an algebraic variety using the cohomology ring of  $A$ . An essential assumption in this theory is the finite generation condition for the cohomology ring of  $A$  and that for the corresponding modules.

In this paper, we introduce various approaches to study the finite generation condition. First, we show that the finite generation condition on  $A$ -modules can be replaced by a condition on any affine commutative Hopf  $A$ -module algebra  $R$  under the assumption that  $R$  is integral over its invariant subring  $R^A$ . Next, we use a spectral sequence argument to show that a finite generation condition holds for certain filtered, smash and crossed product algebras in positive characteristic if the related spectral sequences collapse at some pages. Finally, for any finite dimensional Hopf algebra  $A$  that is defined over a number field over the rationals, we construct another finite dimensional Hopf algebra  $A'$  over a finite field and prove that if the finite generation condition holds for  $A'$ , then the same condition holds for  $A$ .

## CONTENTS

1. Introduction	2
2. Preliminaries	4
2.1. Finite generation condition on Hochschild cohomology	4
2.2. Properties of graded algebras	5
2.3. Integral property of Hopf algebras	5
2.4. Extensions for the finite generation conditions	5
3. On the equivalency of finite generation conditions for Hopf algebras	7
4. A spectral sequence argument for the finite generation conditions	10
5. A lifting method for the finite generation conditions via reduction modulo $p$	18
Appendix A. Some relevant spectral sequences related to cohomology	23
A.1. Cohomology of a filtered augmented algebra	23
A.2. Hochschild cohomology of a filtered algebra	24
A.3. Cohomology of an augmented crossed product	25
A.4. Hochschild cohomology of a smash product	27
Acknowledgments	29
References	29

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## 1. INTRODUCTION

Hochschild cohomology was introduced by Hochschild in 1945 [22] for any associative algebra. Gerstenhaber [18] showed that this cohomology has a graded algebra structure (via cup product) and a graded Lie algebra structure (via a Lie bracket or Gerstenhaber bracket). These two algebraic structures are compatible in such a way that makes Hochschild cohomology a *Gerstenhaber algebra*. Many mathematicians have since investigated Hochschild cohomology  $\mathrm{HH}^*(A)$  for various types of algebras  $A$ , and it has been useful in many settings, including algebraic deformation theory (e.g. [19]) and support variety theory (e.g. [10, 46]).

Generally speaking, the theory of support varieties studies representations of an algebra  $A$  geometrically by associating each finitely generated  $A$ -module  $M$  to a certain algebraic variety  $V(M)$ , namely the variety of the kernel of the graded ring homomorphism  $-\otimes_A M : \mathrm{HH}^*(A) \rightarrow \mathrm{Ext}_A^*(M, M)$ . In using Hochschild cohomology in support variety theory, the following finite generation assumption on the Hochschild cohomology ring  $\mathrm{HH}^*(A)$  of  $A$  and on bimodules is essential (see e.g., [10, 46, 47]).

**(fg)**:  $\mathrm{HH}^*(A, M)$  is a noetherian module over  $\mathrm{HH}^*(A)$ , for any finitely generated  $A$ -bimodule  $M$ .

In particular, condition **(fg)** implies that the Hochschild cohomology ring  $\mathrm{HH}^*(A)$  is finitely generated. Note that this is not always true for a finite dimensional algebra  $A$ ; see Xu's counterexample in [54]. We wish to know more finite dimensional algebras  $A$  that satisfy condition **(fg)**. There are several conditions equivalent to **(fg)**, which are sometimes more convenient to use. For completeness, we include them in Proposition 2.3 below.

The theory of support varieties originated in group representation theory. In the early 1960's, Golod [21], Venkov [51], and Evens [12] proved that the cohomology ring of a finite group is always finitely generated. This result opened the door to geometric approaches in the study of modular representations of finite groups, pioneered by Quillen [44] and expanded by Carlson [9] and Avrunin-Scott [5]. Later, Friedlander and Suslin [17] extended the finite generation result to finite dimensional cocommutative Hopf algebras (equivalently, finite group schemes), and again promoted the algebro-geometric method.

However, for any finite dimensional Hopf algebra  $A$  over a field  $k$ , it is still unknown in general if its cohomology ring  $H^*(A, k)$  is always finitely generated. In fact, this is a long-standing conjecture, which is formulated in the setting of finite tensor categories by Etingof and Ostrik in [14, Conjecture 2.18]. If this conjecture holds true for  $A$ , one can define support varieties over  $A$  by using a graded ring homomorphism  $-\otimes_k M : H^*(A, k) \rightarrow \mathrm{Ext}_A^*(M, M)$  similar to that described for Hochschild cohomology above. This homomorphism factors through the action of  $\mathrm{HH}^*(A)$  on  $\mathrm{Ext}_A^*(M, M)$  [42], giving a connection between the support variety theories defined via Hopf algebra cohomology  $H^*(A, k)$  and Hochschild cohomology  $\mathrm{HH}^*(A)$ . We will work specifically with the following noetherian assumption of a finite dimensional Hopf  $k$ -algebra  $A$  (see e.g., [16]), and more generally for any finite dimensional augmented  $k$ -algebra  $A$ :

**(hfg)**:  $H^*(A, M)$  is a noetherian module over  $H^*(A, k)$ , for any finite dimensional  $A$ -module  $M$ .

In particular, when  $A$  is a finite dimensional Hopf algebra, condition **(hfg)** implies that  $H^*(A, k)$  is finitely generated, which is the finite generation conjecture mentioned above. We wish to identify more finite dimensional Hopf algebras  $A$  that satisfy condition **(hfg)**, and hence move forwards towards the goal of proving the finite generation conjecture.

**Main Results:** We take three different approaches in this finite generation problem:

Our first result in Section 3 provides several equivalent assumptions for **(hfg)** including the finite generation condition on the cohomology ring  $H^*(A, R)$  for any affine commutative  $A$ -module algebra  $R$ . Our result in Lemma 3.3 shows that these two assumptions are equivalent if  $A$  has the integral property (see Section 2.3). Although there are examples of finite dimensional Hopf algebras that do not possess the integral property (e.g., the 4-dimensional Sweedler Hopf algebra), a classical result in algebraic groups says that any finite dimensional cocommutative Hopf algebra has the integral property. Zhu [55] proved that the integral property holds for any finite dimensional cosemisimple Hopf algebra and Skryabin [45] showed that any finite dimensional Hopf algebra in positive characteristic has the integral property. So our result can be applied to all these cases.

Next, in Section 4, we study the finite generation conditions that are preserved under certain spectral sequences related to filtered, smash and crossed product algebras (see Theorems 4.9, 4.10, and 4.11). For any finite dimensional (Hopf) algebra  $A$ , spectral sequences are powerful tools for handling its cohomology rings (see the appendices for those we use here). In the work of Friedlander and Suslin for finite group schemes [17, Lemma 1.6], an answer to the finite generation conditions on multiplicative spectral sequences relies on having suitable *permanent cocycles* (those universal cocycles which survive under the differential maps of all pages in a cohomology spectral sequence). A way of finding such permanent cocycles first came from Evens' work of proving finite generation of group cohomology, using the *Evens norm map* [13, §6] and the Lyndon-Hochschild-Serre spectral sequence. In practice, specific resolutions are used to identify the needed permanent cocycles (see e.g., [28, 39, 41]). In this work, we use the Frobenius map to construct such permanent cocycles over the field of positive characteristic (see Lemma 4.8). In Propositions 4.6 and 4.7, we employ the Hilton-Eckmann argument in a suspended monoidal category to show that the cohomology ring  $H^*(A, R)$  of a finite dimensional Hopf algebra  $A$  with coefficients in an  $A$ -module algebra  $R$  is finite over its affine graded center under some assumptions on  $R$ . We apply this result to the first pages of May spectral sequences related to filtered algebras and to the second pages of Lyndon-Hochschild-Serre spectral sequences related to smash and crossed products. Moreover, if the related spectral sequences collapse over a field of positive characteristic (that is, the differential maps in such cohomology spectral sequences become zero after certain pages), then we are able to construct permanent cocycles by using the Frobenius map on their graded centers to conclude the finite generation conditions.

Finally, in Section 5, we apply the well-known reduction modulo  $p$  method in number theory to support variety theory. It is based on the following observation that for a large family of existing examples of finite dimensional complex Hopf algebras (e.g., Lusztig's small quantum groups [27]) they can all be defined over some algebraic number field  $K$  over the rational numbers  $\mathbb{Q}$ . So by "clearing the denominators", they further can be defined as Hopf algebras over some localizations of the ring of integers  $\mathcal{O}_K$ . These facts suggest us to select a "good" prime  $p$  and to construct from each such complex Hopf algebra  $A$  another finite dimensional Hopf algebra  $A'$  over the finite field  $F_q$ , where  $q = p^n$  for some integer  $n$ . We show that if the newly constructed Hopf algebra  $A'$  over the finite field satisfies **(hfg)** then so does the original complex Hopf algebra  $A$ . Namely, the finite generation conditions in positive characteristic can be lifted up to those in zero characteristic via reduction modulo  $p$  (see Theorem 5.5). Moreover, our approach will not only be limited to the cohomology ring of finite dimensional Hopf algebras but will also be applicable to the Hochschild cohomology of finite dimensional associative algebras (see Theorem 5.6). Therefore, our lifting method reduces the study of the finite generation conditions for finite dimensional (Hopf) algebras over an arbitrary field to those only in positive characteristic where one might apply certain characteristic  $p$  techniques

to settle the finite generation conjecture affirmatively. A unique aspect of this work is to provide a viable connection between a field of *characteristic zero* and a field of *positive characteristic* in studying the cohomology of Hopf algebras. This link is often lacking when one classifies Hopf algebras, as the classification methods are quite different from one field to the other (see e.g., [1, 36, 37, 38, 52]).

## 2. PRELIMINARIES

Throughout the paper, let  $k$  be a base field. All modules are left modules and  $\otimes = \otimes_k$  unless stated otherwise. We first recall the two cohomology types discussed in this paper and present some background material necessary to build up our main results.

**Definition 2.1.** Let  $A$  be an algebra over  $k$  and  $M$  be an  $A$ -bimodule, which can be considered as a left module over the enveloping algebra  $A^e = A \otimes A^{op}$  of  $A$ . The **Hochschild cohomology** of  $A$  with coefficients in  $M$  is

$$\mathrm{HH}^*(A, M) := \bigoplus_{n \geq 0} \mathrm{Ext}_{A^e}^n(A, M),$$

where  $A$  is an  $A$ -bimodule via the left and right multiplications in  $A$ .

Under the cup product,  $\mathrm{HH}^*(A) := \mathrm{HH}^*(A, A)$  is a graded commutative  $k$ -algebra and  $\mathrm{HH}^*(A, M)$  is a module over  $\mathrm{HH}^*(A)$ .

**Definition 2.2.** Let  $A$  be an augmented  $k$ -algebra and  $M$  be a left  $A$ -module. The **cohomology** of  $A$  with coefficients in  $M$  is

$$\mathrm{H}^*(A, M) := \bigoplus_{n \geq 0} \mathrm{Ext}_A^n(k, M),$$

where  $k$  is an  $A$ -module via the augmentation map.

Under the Yoneda product,  $\mathrm{H}^*(A, k)$  is a graded  $k$ -algebra and  $\mathrm{H}^*(A, M)$  is a module over  $\mathrm{H}^*(A, k)$ . If, in addition,  $A$  is a finite dimensional Hopf algebra, then  $\mathrm{H}^*(A, k)$  is graded commutative (see also e.g., [20, 50]).

**2.1. Finite generation condition on Hochschild cohomology.** We consider the following finite generation conditions on Hochschild cohomology as alternatives to the **(fg)** condition:

- (fg1):** There is a finitely generated commutative graded subalgebra  $H$  of  $\mathrm{HH}^*(A)$  with degree-0 component  $H_0 = \mathrm{HH}^0(A)$ ; and
- (fg2):**  $\mathrm{Ext}_A^*(M, N)$  is a finitely generated  $H$ -module, for all pairs of finitely generated  $A$ -modules  $M$  and  $N$ .

We next recall how these finite generation conditions and others are all equivalent.

**Proposition 2.3.** *For any finite dimensional algebra  $A$ , the following are equivalent.*

- (i)  $\mathrm{HH}^*(A, M)$  is a noetherian module over  $\mathrm{HH}^*(A)$ , for any finite  $A$ -bimodule  $M$ , that is,  $A$  satisfies **(fg)**.
- (ii)  $\mathrm{HH}^*(A)$  is a finitely generated algebra and  $\mathrm{Ext}_A^*(A/\mathrm{rad}(A), A/\mathrm{rad}(A))$  is a finitely generated module over  $\mathrm{HH}^*(A)$ , where  $\mathrm{rad}(A)$  denotes the radical of  $A$ .
- (iii)  $\mathrm{HH}^*(A)$  is a finitely generated algebra and  $\mathrm{Ext}_A^*(M, N)$  is a finitely generated module over  $\mathrm{HH}^*(A)$ , for all pairs of finite  $A$ -modules  $M$  and  $N$ .
- (iv)  $A$  satisfies **(fg1)** and **(fg2)**.

*Proof.* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii): The proof is essentially that of [10, Proposition 2.4]. Since  $\mathrm{HH}^*(A)$  is graded commutative, we know it is finitely generated if and only if it is noetherian by [10, Proposition 2.4(2)]. Also, a module over a noetherian ring is finitely generated if and only if it is a noetherian module.

(ii) $\Leftrightarrow$ (iv): See [47, Proposition 5.7].  $\square$

**2.2. Properties of graded algebras.** Both cohomology structures  $\mathrm{HH}^*(A)$  and  $\mathrm{H}^*(A, k)$  are graded algebras, so to work with these cohomology structures, we recall some basic properties of nonnegatively graded algebras.

Let  $A$  be a (nonnegatively) graded  $k$ -algebra, that is,  $A = \bigoplus_{i \geq 0} A_i$  with  $1_A \in A_0$  and  $A_i A_j \subseteq A_{i+j}$  for all integers  $i, j \geq 0$ . If  $A_0 = k$ , then  $A$  is called **connected graded**. Recall that  $A$  is called  **$A_0$ -affine** if it is a finitely generated algebra over the degree-0 component  $A_0$ . The following well-known results will be used many times throughout the paper without explicit citation.

**Proposition 2.4.** *Let  $A = \bigoplus_{i \geq 0} A_i$  be a graded algebra for which the subalgebra  $A_0$  is noetherian commutative. Then:*

- (i)  *$A$  is noetherian if and only if  $A$  is graded noetherian.*
- (ii) *Suppose  $A$  is graded commutative. Then the following are equivalent:*
  - (a)  *$A$  is noetherian.*
  - (b)  *$A$  is  $A_0$ -affine.*
  - (c)  *$A$  is a finite module over  $A^{\mathrm{ev}}$  and  $A^{\mathrm{ev}}$  is  $A_0$ -affine, where  $A^{\mathrm{ev}}$  consists of all even-degree elements of  $A$ .*
- (iii) *Suppose  $A$  is noetherian and is a finite module over some graded central subalgebra  $Z$  with  $Z_0 = A_0$ . Then  $Z$  is  $Z_0$ -affine and noetherian.*

*Proof.* For a proof of (i), see for example [12]. For (ii), see [23, Lemma 3.2]. For (iii), we can adapt the Artin-Tate Lemma [32, Lemma 13.9.10] to the graded setting.  $\square$

**2.3. Integral property of Hopf algebras.** We recall that a commutative ring  $R$  is **integral** over some subring  $T$  if every element of  $R$  is a root of some monic polynomial with coefficients in  $T$ . Furthermore, if  $R$  is  $k$ -affine, then  $R$  is integral over  $T$  if and only if  $R$  is a finite module over  $T$ . We say a Hopf algebra  $A$  has the **integral property** if for any commutative  $A$ -module algebra  $R$ , it is integral over its invariant subring  $R^A$ . Note that, in characteristic zero, Zhu [55] showed that the four dimensional Sweedler Hopf algebra does *not* have the integral property.

**Proposition 2.5.** *Let  $A$  be a finite dimensional Hopf algebra over a field  $k$ . In each of the following cases,  $A$  satisfies the integral property:*

- (i) *the base field  $k$  has positive characteristic,*
- (ii)  *$A$  is semisimple,*
- (iii)  *$A$  is cosemisimple,*
- (iv)  *$A$  is cocommutative.*

*Proof.* (i) is [45, Proposition 2.7 and the remark below it]. (ii) is [45, Theorem 6.1 and the remark below it]. (iii) follows from [55, Theorem 2.1]. (iv) is [33, Theorem 4.2.1].  $\square$

**2.4. Extensions for the finite generation conditions.** In this section, we remark that the finite generation conditions **(fg)** and **(hfg)** that were defined on (Hochschild) cohomology of an algebra  $A$  in the introduction can be extended to another algebra  $B$  under some assumptions:

**Lemma 2.6.** *Let  $f : A \rightarrow B$  be a map of finite dimensional augmented algebras such that  $B$  is projective as a (right)  $A$ -module via  $f$ . If  $B$  satisfies **(hfg)**, then so does  $A$ .*

*Proof.* Denote by  $\mathcal{C} = \text{Mod}(A)$  and  $\mathcal{D} = \text{Mod}(B)$  the categories of left  $A$ -modules and left  $B$ -modules, respectively. Then there is a natural adjoint pair of functors

$$\mathcal{F} = {}_B B_A \otimes_A - : \mathcal{C} \rightarrow \mathcal{D} \quad \text{and} \quad \mathcal{G} = \text{Hom}_B({}_B B_A, -) : \mathcal{D} \rightarrow \mathcal{C}.$$

Specifically,  $\mathcal{F}$  is tensor induction and  $\mathcal{G}$  takes a  $B$ -module  $M$  to the  $A$ -module  $\text{Hom}_B({}_B B_A, M)$  on which the  $A$ -module structure is given by  $(a \cdot f)(b) = f(ba)$  for all  $a \in A$ ,  $b \in B$ , and  $f \in \text{Hom}_B({}_B B_A, M)$ . Since  $B$  satisfies **(hfg)**, we know  $\text{Ext}_B^*(k, k)$  is left noetherian and  $\text{Ext}_B^*(k, B \otimes_A k)$  is finitely generated over  $\text{Ext}_B^*(k, k)$ . Since  $B_A$  is projective, functor  $\mathcal{G}$  is exact and preserves injective objects. Note that  $\mathcal{G}(k) = k$  and  $\mathcal{F}\mathcal{G}(k) = \mathcal{F}(k) = B \otimes_A k$ . By [11, Proposition 9.2(d)], we know  $\text{Ext}_A^*(k, k)$  is left noetherian. Moreover, for any finite dimensional  $A$ -module  $M$ ,  $\text{Ext}_B^*(k, \mathcal{F}(M))$  is finitely generated over  $\text{Ext}_B^*(k, k)$ . Using a right module version of [11, Proposition 9.2(c)], we have  $\text{Ext}_A^*(k, M)$  is finitely generated over  $\text{Ext}_A^*(k, k)$ . So  $A$  satisfies **(hfg)**.  $\square$

Applying a similar argument, we can prove the following lemma about the finite generation condition for Hochschild cohomology.

**Lemma 2.7.** *Let  $f : A \rightarrow B$  be a map of finite dimensional algebras such that  $B$  is projective both as left and right  $A$ -modules via  $f$ . If  $B$  satisfies **(fg)**, then so does  $A$ .*

The next result was first proved in [35] in the context of finite tensor categories with respect to surjective tensor functors. We provide here an alternate proof in the special case of categories of modules over Hopf algebras.

**Proposition 2.8.** *Let  $A \subset B$  be an extension of finite dimensional Hopf algebras. If  $B$  satisfies **(hfg)**, then so does  $A$ .*

*Proof.* By the Nichols-Zoeller freeness theorem [40], every finite dimensional Hopf algebra is free as a module over each of its Hopf subalgebras. The statement now follows from Lemma 2.6.  $\square$

Next we discuss the relations between conditions **(fg)** and **(hfg)** for a finite dimensional Hopf algebra  $A$ . Let  $S$  denote the antipode map of  $A$ .

**Proposition 2.9.** *Let  $A$  be a finite dimensional Hopf algebra. Then  $A$  satisfies **(hfg)** if and only if  $A$  satisfies **(fg)** and  $\text{HH}^*(A)$  is a finitely generated module over  $\text{H}^*(A, k)$ .*

*Proof.* Denote by  $\text{Mod}(A)$  and  $\text{Mod}(A^e)$  the categories of left  $A$ -modules and  $A$ -bimodules, respectively. There is a natural pair of adjoint functors

$$\mathcal{F} = A^e \otimes_A - : \text{Mod}(A) \rightarrow \text{Mod}(A^e) \quad \text{and} \quad \mathcal{G} = \text{Hom}_{A^e}(A^e A_A^e, -) : \text{Mod}(A^e) \rightarrow \text{Mod}(A),$$

where  $A^e$  is viewed as a right  $A$ -module via the embedding  $\delta : A \rightarrow A^e$  such that  $\delta(a) = \sum a_1 \otimes S(a_2)$  for any  $a \in A$ . Note that  $A^e$  is a free right  $A$ -module by the fundamental theory of Hopf modules (e.g., [26, Lemma 2.3(ii)]). So  $\mathcal{F}$  and  $\mathcal{G}$  are both exact and preserve projective and injective objects, respectively. And there is an isomorphism of  $A^e$ -modules  $A \cong A^e \otimes_A k = \mathcal{F}(k)$  [42, Lemma 7.1].

Suppose  $A$  satisfies **(hfg)**. Apply [11, Proposition 9.1(c)] where  $C = k$ ,  $\mathcal{F}(k) = A$  and  $D = M$  as any finite  $A$ -bimodule. Then  $\text{HH}^*(A, M)$  is finitely generated over  $\text{HH}^*(A)$  since  $\text{H}^*(A, \mathcal{G}(M))$  is finitely generated over  $\text{H}^*(A, k)$  by **(hfg)**. Now letting  $D = A$ , Hochschild cohomology  $\text{HH}^*(A) = \text{H}^*(A, \mathcal{G}(A))$  is finitely generated over  $\text{H}^*(A, k)$ . Moreover, Proposition [11, Proposition 9.1(d)] implies that  $\text{HH}^*(A)$  is noetherian.

Conversely, we can view any finite  $A$ -module  $M$  as a bimodule over  $A$  by equipping it with the trivial right action, which is denoted by  $M^{\text{tr}}$ . It is clear that  $\mathcal{F}(M^{\text{tr}}) = M$ . By **(fg)**, we know  $\text{HH}^*(A, M^{\text{tr}}) \cong \text{H}^*(A, M)$  is finitely generated over  $\text{HH}^*(A)$ . The action of  $\text{H}^*(A, k)$  factors through

that of  $\mathrm{HH}^*(A)$  [42, Lemma 7.3] and  $\mathrm{HH}^*(A)$  is a finitely generated module over  $\mathrm{H}^*(A, k)$ . Hence  $\mathrm{H}^*(A, M)$  is finitely generated over  $\mathrm{H}^*(A, k)$ . Proposition 2.4(3) implies that  $\mathrm{H}^*(A, k)$  is noetherian. So  $A$  satisfies **(hfg)**.  $\square$

### 3. ON THE EQUIVALENCY OF FINITE GENERATION CONDITIONS FOR HOPF ALGEBRAS

In this section, we provide some equivalent descriptions of the finite generation conditions on the cohomology of any finite dimensional Hopf algebra. First of all, we assert that the assumption **(hfg)** is preserved under any field extension:

**Lemma 3.1.** *Let  $A$  be a finite dimensional Hopf algebra over a field  $k$ , and let  $K$  be any field extension of  $k$ . Then  $A$  satisfies **(hfg)** if and only if  $A' = A \otimes_k K$  satisfies **(hfg)**.*

*Proof.* Suppose  $A'$  satisfies **(hfg)**. Let  $M$  be a finite module over  $A$ , and  $M' = M \otimes_k K$  the corresponding finite module over  $A'$ . Since  $K$  is flat over  $k$ , there is a graded ring isomorphism  $\mathrm{H}^*(A', K) \cong \mathrm{H}^*(A, k) \otimes_k K$ . Hence there is an embedding of categories from  $\mathrm{H}^*(A, k)$ -modules to  $\mathrm{H}^*(A', K)$ -modules given by the tensor product  $- \otimes_k K$ . By hypothesis,  $\mathrm{H}^*(A', K)$  is noetherian, and it now follows that  $\mathrm{H}^*(A, k)$  is noetherian.

It remains to show that  $\mathrm{H}^*(A, M)$  is finitely generated over  $\mathrm{H}^*(A, k)$ . It suffices to show that  $\mathrm{H}^*(A, M)$  is finitely presented. By a result of Lenzing [25, Satz 3], it is equivalent to show that  $\mathrm{Hom}_{\mathrm{H}^*(A, k)}(\mathrm{H}^*(A, M), -)$  preserves any inductive limit  $\varinjlim M_i$  in the category of  $\mathrm{H}^*(A, k)$ -modules. There is a natural map

$$\varinjlim \mathrm{Hom}_{\mathrm{H}^*(A, k)}(\mathrm{H}^*(A, M), M_i) \xrightarrow{f} \mathrm{Hom}_{\mathrm{H}^*(A, k)}(\mathrm{H}^*(A, M), \varinjlim M_i).$$

After applying  $- \otimes_k K$ , it becomes

$$\varinjlim \mathrm{Hom}_{\mathrm{H}^*(A', K)}(\mathrm{H}^*(A', M'), M'_i) \xrightarrow{f \otimes_k K} \mathrm{Hom}_{\mathrm{H}^*(A', K)}(\mathrm{H}^*(A', M'), \varinjlim M'_i),$$

where  $M'_i = M_i \otimes_k K$ . The map  $f \otimes_k K$  is an isomorphism since  $\mathrm{H}^*(A', M')$  is finitely generated over the noetherian algebra  $\mathrm{H}^*(A', K)$  and hence it is finitely presented and preserves the inductive limit  $\varinjlim M'_i$  in the category of  $\mathrm{H}^*(A', K)$ -modules. This implies that  $f$  is an isomorphism, since  $- \otimes_k K$  is exact.

On the other hand, suppose  $A$  satisfies **(hfg)**. We first deal with the case when  $K/k$  is a finite field extension. By hypothesis and by the graded commutativity of  $\mathrm{H}^*(A, k)$ , it is easy to see that  $\mathrm{H}^*(A', K) \cong \mathrm{H}^*(A, k) \otimes_k K$  is noetherian and  $K$ -affine. Let  $M'$  be a finite module over  $A'$ , and  $M = {}_A M'$  be the restriction of  $M'$  to  $A$ , which is again a finite  $A$ -module since  $A' = A \otimes_k K$  is a free  $A$ -module of rank  $[K : k]$ . Then  $\mathrm{H}^*(A, M)$  is finitely generated over  $\mathrm{H}^*(A, k)$  since  $A$  satisfies **(hfg)**. Moreover, by the tensor-hom adjoint pair, there are isomorphisms of graded vector spaces:

$$\begin{aligned} \mathrm{H}^*(A, M) &\cong \mathrm{Ext}_A^*(k, \mathrm{Hom}_{A'}({}_A A', M')) \\ &\cong \mathrm{Ext}_{A'}^*(A' \otimes_A k, M') \\ &\cong \mathrm{Ext}_{A'}^*(K, M') = \mathrm{H}^*(A', M'). \end{aligned}$$

Thus,  $\mathrm{H}^*(A', M')$  is finitely generated over  $\mathrm{H}^*(A', K) = \mathrm{H}^*(A, k) \otimes_k K$ . This implies that  $A'$  satisfies **(hfg)** whenever  $K/k$  is a finite field extension.

In general, say  $k \subset K$  is any field extension. By the previous discussion, it suffices to show that  $A' = A \otimes_k \overline{K}$  satisfies **(hfg)**, where  $\overline{K}$  is the algebraic closure of  $K$ . We can argue similarly that  $\mathrm{H}^*(A', \overline{K}) = \mathrm{H}^*(A, k) \otimes_k \overline{K}$  is noetherian and finitely generated. Since  $A$  is finite dimensional, there exists a finite field extension  $F$  of  $k$  such that the quotient  $A \otimes_k F / \mathrm{rad}(A \otimes_k F)$  is a direct

sum of matrix algebras over  $F$  (for instance, we can take  $F$  to be the splitting field of  $A/\text{rad}(A)$ ). Now let  $B = A \otimes_k F$ , so  $B \otimes_F \overline{K} = A'$  and  $(B/\text{rad}(B)) \otimes_F \overline{K} \cong A'/\text{rad}(A')$ . Since  $F/k$  is a finite field extension,  $B$  satisfies **(hfg)** by the previous discussion. Therefore,  $H^*(B, B/\text{rad}(B))$  is finitely generated over  $H^*(B, F)$  and

$$H^*(B, B/\text{rad}(B)) \otimes_F \overline{K} \cong H^*(B \otimes_F \overline{K}, (B/\text{rad}(B)) \otimes_F \overline{K}) \cong H^*(A', A'/\text{rad}(A'))$$

is finitely generated over  $H^*(A', \overline{K}) = H^*(B, F) \otimes_F \overline{K}$ . This implies that  $H^*(A', S)$  is finitely generated over  $H^*(A', \overline{K})$  for any  $A'$ -simple  $S$ . Finally, by filtering any finite dimensional  $A'$ -module by its composition series, we see that  $A'$  satisfies **(hfg)**.  $\square$

*Remark 3.2.* Using a similar argument, we can show that for any finite dimensional (not necessarily Hopf) algebra  $A$  over a field  $k$ , and any field extension  $K$  of  $k$ ,  $A$  satisfies the Hochschild condition **(fg)** if and only if  $A' = A \otimes_k K$  satisfies **(fg)**.

In the classical case of group cohomology, it is well-known that the group algebra  $kG$  of any finite group  $G$  satisfies **(hfg)**. More generally, Evens proved in [12, Theorem 8.1] that for any affine commutative algebra  $R$  which admits a  $kG$ -module algebra structure,  $H^*(G, M) := H^*(kG, M)$  is a finitely generated module over  $H^*(G, R)$  for any finitely generated module  $M$  over the skew group ring  $R \# kG$ . This nice property of finite group cohomology prompts us to seek an analogy in the reign of finite dimensional Hopf algebras. Thus we consider the following generalized noetherian assumption on a finite dimensional Hopf algebra  $A$ , where  $R \# A$  denotes a smash product algebra (see [33]):

**(hfg\*)**:  $H^*(A, M)$  is a noetherian module over  $H^*(A, R)$  for any affine commutative  $A$ -module algebra  $R$  and any finitely generated  $R \# A$ -module  $M$ .

It is clear that **(hfg\*)** implies **(hfg)** by taking  $R = k$ . The converse implication will also be true under an additional assumption, namely the integral property of  $A$  (see Section 2.3) as we show in the following lemma.

**Lemma 3.3.** *Let  $A$  be a finite dimensional Hopf algebra. Suppose  $A$  has the integral property. Then the following are equivalent.*

- (i)  $A$  satisfies **(hfg)**.
- (ii)  $A$  satisfies **(hfg\*)**.
- (iii)  $H^*(A, R)$  is finitely generated for any affine commutative  $A$ -module algebra  $R$ .
- (iv) Let  $T = \bigoplus_{i \geq 0} T_i$  be a finitely generated graded noetherian  $A$ -module algebra that is a finite module over some graded central  $A$ -module subalgebra  $Z$  of  $T$ , and let  $M$  be a finitely generated module over  $T \# A$ . Then  $H^*(A, M)$  is noetherian over  $H^*(A, T)$ .

*Proof.* (i) $\Rightarrow$ (ii): Without loss of generality, by Lemma 3.1, we can replace  $k$  by a finite field extension and assume  $A/\text{rad}(A)$  is a direct sum of matrix algebras over  $k$ . Let  $R$  be an affine commutative  $A$ -module algebra.

We first treat the case when  $A$  acts on  $R$  trivially such that  $R \# A \cong R \otimes A$ . Let  $M$  be an  $R \otimes A$ -module. If  $M$  is cyclic, we can write  $M = (R \otimes A)/I$  for some left ideal  $I$  of  $R \otimes A$ . Since  $A$  is finite dimensional, there is a composition series  $0 = V_0 \subset V_1 \subset V_2 \cdots \subset V_n = A$  of left  $A$ -modules, where each factor  $S_i = V_i/V_{i-1}$  is a simple  $A$ -module. This induces a finite filtration  $0 = M_0 \subset M_1 \subset M_2 \cdots \subset M_n = M$  on  $M$  after applying  $R \otimes_k -$ , where each factor  $M_i/M_{i-1}$  is a quotient module of  $R \otimes S_i$ . Note that  $R \otimes S_i$  is a cyclic module over  $R \otimes (A/\text{Ann}(S_i)) \cong R \otimes M_{d_i}(k) \cong M_{d_i}(R)$ . By replacing  $R \otimes A$  with  $M_{d_i}(R)$ , one sees that  $R \otimes S_i \subset M_{d_i}(R)$  is just column matrices  $R^{d_i}$ . Then by some matrix multiplication,  $M_i/M_{i-1} \cong (R/J_i) \otimes S_i$  for some ideal  $J_i$  of  $R$ . Since  $A$  acts

on  $R$  trivially, by the universal coefficients theorem,  $H^*(A, R) \cong R \otimes H^*(A, k)$ , and so is finitely generated and noetherian. Moreover,  $H^*(A, (R/J_i) \otimes S_i) = (R/J_i) \otimes H^*(A, S_i)$  is finitely generated over  $H^*(A, R) = R \otimes H^*(A, k)$  since  $H^*(A, S_i)$  is finitely generated over  $H^*(A, k)$  by **(hfg)**. For each  $i$ , by applying  $H^*(A, -)$  to the short exact sequence  $0 \rightarrow M_{i-1} \rightarrow M_i \rightarrow (R/J_i) \otimes S_i \rightarrow 0$ , we obtain an exact sequence

$$H^*(A, M_{i-1}) \longrightarrow H^*(A, M_i) \longrightarrow H^*(A, (R/J_i) \otimes S_i),$$

where we have just shown that  $H^*(A, (R/J_i) \otimes S_i)$  is finitely generated over  $H^*(A, R)$ . Then induction on  $i$  yields that  $H^*(A, M_i)$  is finitely generated for all  $i$ . This completes the cyclic case. In general, we can again induct on the number of minimal generators of  $M$  and employ an exact sequence similar to that above to conclude this trivial  $A$ -action case.

Finally, if  $A$  acts on  $R$  arbitrarily, by the integral property of  $A$ ,  $R$  is a finitely generated module over the invariant subring  $R^A$ . Then by the previous discussion,  $H^*(A, M)$  is finitely generated over  $H^*(A, R^A)$  and hence is finitely generated over  $H^*(A, R)$ . In particular, when  $M = R$ , we conclude that  $H^*(A, R)$  is finitely generated as a module over  $H^*(A, R^A) = R^A \otimes H^*(A, k)$ , which is noetherian.

(iii) $\Rightarrow$ (i): By letting  $R = k$ , we know  $H^*(A, k)$  is finitely generated and noetherian. For any finite  $A$ -module  $M$ , denote  $R = k \oplus M$  where  $M^2 = 0$ . Then one can check that  $H^*(A, R)$  is finitely generated implies that  $H^*(A, M)$  is finitely generated over  $H^*(A, k)$ .

(ii) $\Rightarrow$ (iv): Since  $A$  satisfies **(hfg\*)**,  $H^*(A, M)$  is noetherian over  $H^*(A, Z^{\text{ev}})$ . As the action of  $H^*(A, Z^{\text{ev}})$  on  $H^*(A, M)$  factors through that of  $H^*(A, T)$ , it follows that  $H^*(A, M)$  is noetherian over  $H^*(A, T)$ .

(ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (ii) are clear. □

Now we are able to summarize various equivalent finite generation conditions on the cohomology of a finite dimensional Hopf algebra.

**Proposition 3.4.** *The following are equivalent for a finite dimensional Hopf algebra  $A$ .*

- (i)  $A$  satisfies **(hfg)**.
- (ii)  $A$  satisfies **(fg)** and  $\text{HH}^*(A)$  is a finitely generated module over  $H^*(A, k)$ .
- (iii)  $H^*(A, k)$  is a finitely generated algebra and  $\text{Ext}_A^*(A/\text{rad}(A), A/\text{rad}(A))$  is a finitely generated module over  $H^*(A, k)$ .
- (iv)  $H^*(A, k)$  is a finitely generated algebra and  $\text{Ext}_A^*(k, A/\text{rad}(A))$  is a finitely generated module over  $H^*(A, k)$ .
- (v)  $H^*(A, k)$  is a finitely generated algebra and  $\text{Ext}_A^*(M, N)$  is a finitely generated module over  $H^*(A, k)$  for all pairs of finite  $A$ -modules  $M$  and  $N$ .
- (vi)  $H^*(A, k)$  is a finitely generated algebra and  $\text{Ext}_A^*(k, M)$  is a finitely generated module over  $H^*(A, k)$  for any finite  $A$ -module  $M$ .

If in addition  $A$  has the integral property, each of the above conditions is equivalent to **(hfg\*)**.

*Proof.* (i) $\Leftrightarrow$ (ii): is Proposition 2.9.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v): These implications follow directly from [10, Proposition 1.4] (see Proposition 2.3), where we take  $H^*(A, k)$  as a subalgebra of  $\text{HH}^*(A) \cong H^*(A, A^{\text{ad}}) = H^*(A, k) \oplus H^*(A, I)$  [42, Lemma 7.2]. Here  $A^{\text{ad}}$  is the adjoint representation of  $A$  and  $I$  is the augmentation ideal of  $A$ . Note that in each case,  $\text{HH}^*(A)$  is a finitely generated module over  $H^*(A, k)$  and hence it is noetherian.

(i) $\Leftrightarrow$ (iv): Direction “ $\Rightarrow$ ” is clear and the other direction comes from filtering a finite  $A$ -module by its composition series and noting that (iv) implies  $H^*(A, S)$  is finite generated over  $H^*(A, k)$  for any simple  $A$ -module  $S$ .

(i) $\Leftrightarrow$ (vi): It is clear since  $H^*(A, k)$  is graded commutative.

Finally, the last statement is a consequence of Lemma 3.3.  $\square$

*Remarks 3.5.*

- (a) It is straightforward to check that the condition **(hfg)** is equivalent to the original finite generation condition in [16] which in characteristic  $\neq 2$  says that  $H^{\text{ev}}(A, k)$  is finitely generated, and that for any pair of finite  $A$ -modules  $M$  and  $N$ ,  $\text{Ext}_A^*(M, N)$  is finitely generated over  $H^{\text{ev}}(A, k)$ .
- (b) More generally, **(hfg)** can be stated for any augmented algebra  $A$ , and one can show that (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi) in Proposition 3.4 by further requiring  $H^*(A, k)$  to be noetherian in (vi).

The equivalent finite generation conditions in Proposition 3.4 allow us to study the finite generation of (Hochschild) cohomology from various perspectives.

#### 4. A SPECTRAL SEQUENCE ARGUMENT FOR THE FINITE GENERATION CONDITIONS

In this section, we use the Hilton-Eckmann argument in a suspended monoidal category to prove that the cohomology ring of  $A$  with coefficients in an  $A$ -module algebra  $R$  is finite over its affine graded center under some assumptions of  $R$ . As applications, we show the finite generation conditions hold for certain filtered, smash and crossed product algebras using the corresponding May and Lyndon-Hochschild-Serre spectral sequences.

We start with a special case when the smash product is taken with a finite dimensional cocommutative Hopf algebra. We follow the argument first used in [6, Corollary 3.2.2] for the key step to show that a particular subalgebra lies in the graded center.

**Lemma 4.1.** *Let  $A$  be a finite dimensional cocommutative Hopf algebra satisfying **(hfg)** and let  $R = \bigoplus_{i \geq 0} R_i$  be a graded  $A$ -module algebra. If  $R$  is a finitely generated noetherian algebra that is a finite module over a graded central  $A$ -module subalgebra  $Z$ , then  $H^*(A, R) = \bigoplus_{i+j \geq 0} H^i(A, R_j)$  is a finitely generated noetherian algebra and is a finite module over its graded center (grading by total degree).*

*Proof.* By Proposition 2.4(2) and (3),  $Z$  and  $Z^{\text{ev}}$  are finitely generated and noetherian. By Proposition 2.5(4),  $Z^{\text{ev}}$  is a finitely generated module over its invariant subring  $(Z^{\text{ev}})^A$  and hence so is  $R$ . Write  $W = (Z^{\text{ev}})^A = \bigoplus_{i \geq 0} W_i$ . Note that  $H^*(A, W) \cong H^*(A, k) \otimes W$  is finitely generated noetherian. By Lemma 3.3,  $H^*(A, R)$  is a finitely generated module over  $H^*(A, W)$ . It remains to show that  $H^*(A, W)$  has image in the graded center of  $H^*(A, R)$ , which follows from the commutative diagram:

$$\begin{array}{ccccc}
 & & P_m \otimes P_n & \xrightarrow{(-1)^{\alpha n} f_{m,\alpha} \otimes g_{n,\beta}} & W_\alpha \otimes R_\beta & \xrightarrow{m_R} & R_{\alpha+\beta} \\
 & \nearrow \Delta & \downarrow (-1)^{mn} \tau & & \downarrow (-1)^{mn+n\alpha+\beta m} \tau & & \uparrow m_R \\
 P & & & & & & \\
 & \searrow \Delta' & P_n \otimes P_m & \xrightarrow{(-1)^{\beta m} g_{n,\beta} \otimes f_{m,\alpha}} & R_\beta \otimes W_\alpha & \xrightarrow{(-1)^{(m+\alpha)(n+\beta)}} & R_\beta \otimes W_\alpha
 \end{array}$$

where  $P_\bullet$  is a projective resolution of  $k$  over  $A$ ,  $\Delta : P_\bullet \rightarrow P_\bullet \otimes P_\bullet$  is a diagonal map,  $\Delta'$  is another diagonal map (defined by commutativity of the left triangle),  $f_{m,\alpha} \in H^m(A, W_\alpha)$ ,  $g_{n,\beta} \in H^n(A, W_\beta)$ ,  $\tau$  is the twisting map (which is always a morphism of  $A$ -modules since  $A$  is cocommutative), and  $m_R$  is the multiplication in  $R$ . The signs  $(-1)^{\alpha n}$  and  $(-1)^{\beta m}$  on the rows of the first square come from a standard sign convention. The second square demonstrates the fact that  $W$  lies in the graded center of  $R$ . It follows from the commutativity of the diagram that

$$f_{m,\alpha} \smile g_{n,\beta} = (-1)^{(m+\alpha)(n+\beta)} g_{n,\beta} \smile f_{m,\alpha}.$$

□

In general, in order to obtain needed results when  $A$  is an arbitrary finite dimensional Hopf algebra, we follow the Hilton-Eckmann argument in the context of a suspended monoidal category demonstrated in [50]. A good reference for all the terminologies is [15].

**Definition 4.2.** A **suspended monoidal category** is a 9-tuple  $(\mathcal{C}, \otimes, e, a, \ell, r, T, \lambda, \rho)$  such that  $(\mathcal{C}, \otimes, e, a, r, \ell)$  is a monoidal category,  $T : \mathcal{C} \rightarrow \mathcal{C}$  is an automorphism,  $\lambda_{X,Y} : X \otimes TY \rightarrow T(X \otimes Y)$  and  $\rho : TX \otimes Y \rightarrow T(X \otimes Y)$  are isomorphisms of functors  $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  for each pair of objects  $X$  and  $Y \in \text{obj } \mathcal{C}$ , and the following diagrams commute:

$$\begin{array}{ccc} e \otimes TX & \xrightarrow{\ell} & TX \\ \lambda \downarrow & & \downarrow 1 \\ T(e \otimes X) & \xrightarrow{T\ell} & TX \end{array} \quad \begin{array}{ccc} TX \otimes e & \xrightarrow{r} & TX \\ \rho \downarrow & & \downarrow 1 \\ T(X \otimes e) & \xrightarrow{Tr} & TX, \end{array}$$

while the following diagram anti-commutes:

$$\begin{array}{ccc} TX \otimes TY & \xrightarrow{\rho} & T(X \otimes TY) \\ \lambda \downarrow & (-1) & \downarrow T\lambda \\ T(TX \otimes Y) & \xrightarrow{T\rho} & T^2(X \otimes Y). \end{array}$$

Given a suspended monoidal category, as in [50, §1.5 and §1.6], we can inductively define a series of isomorphisms of functors  $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  by

$$\lambda_q : X \otimes T^q Y \rightarrow T^q(X \otimes Y), \quad \rho_p : T^p X \otimes Y \rightarrow T^p(X \otimes Y),$$

for all  $p, q \in \mathbb{Z}$  with  $\lambda_1 = \lambda, \rho_1 = \rho$ , such that the following diagrams commute:

$$\begin{array}{ccc} e \otimes T^q X & \xrightarrow{\ell} & T^q X \\ \lambda_q \downarrow & & \downarrow 1 \\ T^q(e \otimes X) & \xrightarrow{T^q \ell} & T^q X \end{array} \quad \begin{array}{ccc} T^p X \otimes e & \xrightarrow{r} & T^p X \\ \rho_p \downarrow & & \downarrow 1 \\ T^p(X \otimes e) & \xrightarrow{T^p r} & T^p X, \end{array}$$

while the following diagram  $(-1)^{pq}$ -commutes:

$$\begin{array}{ccc} T^p X \otimes T^q Y & \xrightarrow{\rho_p} & T^p(X \otimes T^q Y) \\ \lambda_q \downarrow & (-1)^{pq} & \downarrow T^p \lambda_q \\ T^q(T^p X \otimes Y) & \xrightarrow{T^q \rho_p} & T^{p+q}(X \otimes Y). \end{array}$$

Next, we have isomorphisms  $\sigma : X \otimes e \cong e \otimes X$  and  $\tau : e \otimes X \cong X \otimes e$  satisfying  $\ell\sigma = r$  and  $r\tau = \ell$  for any  $X \in \text{obj } \mathcal{C}$ . Moreover, all the isomorphisms above can be naturally extended from  $e$  to  $Y$  whenever  $Y = \bigoplus e$  is a direct sum of copies of the identity object  $e$ .

Now, let  $R = \bigoplus_{i \in \mathbb{Z}} R_i$  be a graded ring in  $\mathcal{C}$  with product maps  $m_{ij} : R_i \otimes R_j \rightarrow R_{i+j}$  satisfying the associativity axiom:

$$\begin{array}{ccc} (R_i \otimes R_j) \otimes R_k & \xrightarrow{m_{ij} \otimes 1} & R_{i+j} \otimes R_k \\ a_{ijk} \downarrow & & \downarrow m_{(i+j)k} \\ R_i \otimes (R_j \otimes R_k) & & \\ 1 \otimes m_{jk} \downarrow & & \downarrow m_{i(j+k)} \\ R_i \otimes R_{j+k} & \xrightarrow{m_{i(j+k)}} & R_{i+j+k}. \end{array}$$

Moreover, we say  $R$  is **unital** if there is a morphism  $u : e \rightarrow R_0$  satisfying the unit axiom

$$\begin{array}{ccc} e \otimes R_i & \xrightarrow{u \otimes 1} & R_0 \otimes R_i \\ \ell \searrow & & \swarrow m_{0i} \\ & R_i & \end{array} \quad \begin{array}{ccc} R_i \otimes e & \xrightarrow{1 \otimes u} & R_i \otimes R_0 \\ r \searrow & & \swarrow m_{i0} \\ & R_i & \end{array}.$$

Consider the ring

$$E(R) := \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(e, T^i R_j),$$

where the product in  $E(R)$  is given by the following composition for any  $f : e \rightarrow T^p R_\alpha$  and  $g : e \rightarrow T^q R_\beta$ :

$$(4.1) \quad f \cdot g : e \xleftarrow{\ell=r} e \otimes e \xrightarrow{(-1)^{q\alpha} f \otimes g} T^p R_\alpha \otimes T^q R_\beta \xrightarrow{\rho_p} T^p(R_\alpha \otimes T^q R_\beta) \\ \xrightarrow{T^p \lambda_q} T^{p+q}(R_\alpha \otimes R_\beta) \xrightarrow{T^{p+q} m_{\alpha\beta}} T^{p+q} R_{\alpha+\beta}.$$

The sign  $(-1)^{q\alpha}$  above comes from the sign convention when passing  $g$  over  $R_\alpha$ . Notice that if  $R$  is unital, then  $E(e)$  has image as a graded subalgebra of  $E(R)$ . Moreover, we say that  $\bigoplus_{i \in \mathbb{Z}} M_i$  is a graded (bi)module over  $R$  if there are morphisms  $\ell_{ij} : R_i \otimes M_j \rightarrow M_{i+j}$  (resp.  $r_{ji} : M_j \otimes R_i \rightarrow M_{i+j}$ ) satisfying some obvious compatibility conditions. Suppose  $\bigoplus_{i \in \mathbb{Z}} M_i$  is a graded bimodule over  $R = \bigoplus_{i \in \mathbb{Z}} R_i$  with each  $R_i$  being a direct sum of all  $e$ 's. We call  $M$  a **graded symmetric bimodule** over  $R$  if the following diagrams commute:

$$\begin{array}{ccc} R_i \otimes M_j & \xrightarrow{\tau} & M_j \otimes R_i \\ r_{ij} \searrow & & \swarrow \ell_{ji} \\ & M_{i+j} & \end{array} \quad \begin{array}{ccc} M_j \otimes R_i & \xrightarrow{\sigma} & R_i \otimes M_j \\ r_{ji} \searrow & & \swarrow \ell_{ij} \\ & M_{i+j} & \end{array}$$

The following theorem extends Suárez-Álvarez's results [50] on the graded commutativity of the (Hochschild) cohomology ring of a finite dimensional (Hopf) algebra.

**Theorem 4.3.** *Retain the above notations. Let  $M = \bigoplus_{i \in \mathbb{Z}} M_i$  be a graded symmetric bimodule over  $R = \bigoplus_{i \in \mathbb{Z}} R_i$  with each  $R_i$  being a direct sum of all  $e$ 's. Then  $E(M)$  is a graded symmetric bimodule over  $E(R)$ . In particular,  $E(M)$  is a graded symmetric bimodule over  $E(e)$ .*

*Proof.* Take  $f : e \rightarrow T^p M_\alpha$  and  $g : e \rightarrow T^q R_\beta$ . It is straightforward to check that we have the following  $(-1)^{p\beta}$ -commutative diagram:

$$\begin{array}{ccccccc}
 e & \xleftarrow{r} & e \otimes e & \xrightarrow{(-1)^{p\beta} g \otimes f} & T^q R_\beta \otimes T^p M_\alpha & & \\
 g \downarrow & & g \otimes 1 \downarrow & & (-1)^{p\beta} \downarrow & & 1 \otimes 1 \downarrow \\
 T^q R_\beta & \xleftarrow{r} & T^q R_\beta \otimes e & \xrightarrow{1 \otimes f} & T^q R_\beta \otimes T^p M_\alpha & & \\
 1 \downarrow & & \rho_q \downarrow & & \rho_q \downarrow & & \\
 T^q R_\beta & \xleftarrow{T^q r} & T^q(R_\beta \otimes e) & \xrightarrow{T^q(1 \otimes f)} & T^q(R_\beta \otimes T^p M_\alpha) & \xrightarrow{T^q \lambda_p} & T^{p+q}(R_\beta \otimes M_\alpha) \xrightarrow{T^{p+q} \ell_{\beta\alpha}} T^{p+q} M_{\alpha+\beta}
 \end{array}$$

where the right outer boundary represents  $g \cdot f$ . On the other hand, the right outer boundary of the  $(-1)^{q\alpha+pq}$ -commutative diagram below is equal to  $f \cdot g$ :

$$\begin{array}{ccccccc}
 e & \xleftarrow{\ell=r} & e \otimes e & \xrightarrow{(-1)^{q\alpha} f \otimes g} & T^p M_\alpha \otimes T^q R_\beta & & \\
 g \downarrow & & 1 \otimes g \downarrow & & (-1)^{q\alpha} \downarrow & & 1 \otimes 1 \downarrow \\
 T^q R_\beta & \xleftarrow{\ell} & e \otimes T^q R_\beta & \xrightarrow{f \otimes 1} & T^p M_\alpha \otimes T^q R_\beta & \xrightarrow{\rho_p} & T^p(M_\alpha \otimes T^q R_\beta) \\
 1 \downarrow & & \lambda_q \downarrow & & \lambda_q \downarrow & & (-1)^{pq} \downarrow T^p \lambda_q \\
 T^q R_\beta & \xleftarrow{T^q \ell} & T^q(e \otimes R_\beta) & \xrightarrow{T^q(f \otimes 1)} & T^q(T^p M_\alpha \otimes R_\beta) & \xrightarrow{T^q \rho_p} & T^{p+q}(M_\alpha \otimes R_\beta) \xrightarrow{T^{p+q} r_{\alpha\beta}} T^{p+q} M_{\alpha+\beta}
 \end{array}$$

Therefore, the anti-commutativity of the product of  $f$  and  $g$  can be derived from the next  $(-1)^{\alpha\beta}$ -commutative diagram:

$$\begin{array}{ccccccc}
 e & \xleftarrow{g} & T^q R_\beta & \xleftarrow{T^q \ell} & T^q(e \otimes R_\beta) & \xrightarrow{T^q(f \otimes 1)} & T^q(T^p M_\alpha \otimes R_\beta) \xrightarrow{T^q \rho_p} T^{p+q}(M_\alpha \otimes R_\beta) \xrightarrow{T^{p+q} r_{\alpha\beta}} T^{p+q} M_{\alpha+\beta} \\
 1 \downarrow & & 1 \downarrow & & T^q \sigma \downarrow & & T^q \sigma \downarrow & & T^{p+q} \sigma \downarrow & & (-1)^{\alpha\beta} \downarrow & 1 \downarrow \\
 e & \xleftarrow{g} & T^q R_\beta & \xleftarrow{T^q r} & T^q(R_\beta \otimes e) & \xrightarrow{T^q(1 \otimes f)} & T^q(R_\beta \otimes T^p M_\alpha) \xrightarrow{T^q \lambda_p} T^{p+q}(R_\beta \otimes M_\alpha) \xrightarrow{T^{p+q} \ell_{\beta\alpha}} T^{p+q} M_{\alpha+\beta}
 \end{array}$$

where the top row is equal to  $(-1)^{q\alpha+pq} f \cdot g$  and the bottom row equals  $(-1)^{p\beta} g \cdot f$ . Hence  $f \cdot g = (-1)^{(p+\alpha)(q+\beta)} g \cdot f$ . Note that the third square commutes because of the functoriality of  $\sigma$  and the fourth square commutes by commutativity of

$$\begin{array}{ccc}
 T^p X \otimes e & \xrightarrow{\rho_p} & T^p(X \otimes e) \\
 \downarrow r & & \downarrow T^p r \\
 T^p X & \xrightarrow{1} & T^p X \\
 \uparrow \ell & & \uparrow T^p \ell \\
 e \otimes T^p X & \xrightarrow{\lambda_p} & T^p(e \otimes X)
 \end{array}
 \begin{array}{l}
 \sigma \left( \begin{array}{c} \curvearrowright \\ \downarrow \\ \uparrow \\ \curvearrowright \end{array} \right) T^p \sigma
 \end{array}$$

for any object  $X \in \text{obj } \mathcal{C}$ . □

**Corollary 4.4.** *Let  $(\mathcal{C}, \otimes, e, a, \ell, r, T, \lambda, \rho)$  be a suspended monoidal category. Let  $W = \bigoplus_{i \in \mathbb{Z}} W_i$  be a subring of  $R = \bigoplus_{i \in \mathbb{Z}} R_i$  with each  $W_i$  being a direct sum of all  $e$ 's and the following  $(-1)^{ij}$ -commutative diagrams*

$$\begin{array}{ccc} W_i \otimes R_j & \xrightarrow{\tau} & R_j \otimes W_i \\ & \searrow \mu & \swarrow \mu \\ & R_{i+j} & \end{array} \quad \begin{array}{ccc} R_i \otimes W_j & \xrightarrow{\sigma} & W_j \otimes R_i \\ & \searrow \mu & \swarrow \mu \\ & R_{i+j} & \end{array}$$

Then the subring  $E(W)$  has image in the graded center of  $E(R)$ .

**Corollary 4.5.** [50, Theorem 1.7] *Let  $(\mathcal{C}, \otimes, e, a, \ell, r, T, \lambda, \rho)$  be a suspended monoidal category. Set*

$$E(e) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(e, T^i e).$$

If  $f : e \rightarrow T^p e$  and  $g : e \rightarrow T^q e$ , define  $f \cdot g = T^q f \circ g : e \rightarrow T^{p+q} e$ . Then  $E(e)$  is a commutative graded ring.

*Proof.* Let  $R = M = e$  in Theorem 4.3. Note that the product  $f \cdot g = T^q f \circ g : e \rightarrow T^{p+q} e$  defined in the statement coincides with the formula (4.1); see the proof of [50, Theorem 1.7].  $\square$

**Applications.** For finite dimensional (Hopf) algebras, spectral sequences are powerful tools for handling their cohomology rings. Therefore, we are interested in multiplicative spectral sequences satisfying similar finite generation conditions. The spectral sequences we have in mind are May spectral sequences related to filtered algebras and Lyndon-Hochschild-Serre spectral sequences related to smash and crossed products. They are described explicitly in the appendices for completeness. We use these spectral sequences to conclude the finite generation conditions for the original (Hopf) algebras if the initial pages of these spectral sequences satisfy the finite generation conditions and the corresponding spectral sequences collapse at certain pages in positive characteristic.

**Proposition 4.6.** *Let  $A$  be a finite dimensional Hopf algebra,  $M$  be any finite dimensional  $A$ -module, and  $R = \bigoplus_{i \geq 0} R_i$  be a connected graded  $A$ -module algebra. Then*

- (i)  $H^*(A, k)$  maps to the graded center of  $\text{Ext}_A^*(M, M)$ . Moreover, if  $A$  satisfies **(hfg)**, then  $\text{Ext}_A^*(M, M)$  is noetherian and a finite module over its graded center.
- (ii)  $H^*(A, k)$  maps to the graded center of  $H^*(A, R) = \bigoplus_{i, j \geq 0} H^i(A, R_j)$  (with respect to the total degree). Moreover, if  $A$  satisfies **(hfg\*)** and  $R$  is a finitely generated noetherian algebra and a finite module over some graded central  $A$ -module subalgebra, then  $H^*(A, R)$  is noetherian and a finite module over its graded center.

*Proof.* We use Theorem 4.3 where  $\mathcal{C}$  is the left derived category of  $\text{Mod}(A)$  with  $\otimes = \otimes_k$ ,  $e = k$  and  $T = [1]$  is the shift functor.

(i) Let  $V^*$  be the left dual of  $V$  (see e.g. [15, §2.10]). Note that  $\text{Ext}_A^*(V, V) \cong \text{Ext}_A^*(k, V \otimes V^*)$  and the standard actions of  $H^*(A, k)$  on either one correspond under this isomorphism. By Theorem 4.3,  $H^*(A, k)$  maps to the graded center of  $\text{Ext}_A^*(k, V \otimes V^*)$  via the coevaluation map  $\text{coev} : k \rightarrow V \otimes V^*$ . Moreover, if  $A$  satisfies **(hfg)**, then  $\text{Ext}_A^*(k, V \otimes V^*)$  is finitely generated as a module over the noetherian graded central subalgebra given by the image of  $H^*(A, k)$ , and as a consequence it is noetherian.

(ii) Since  $k = R_0$  certainly lies in the graded center of  $R$ ,  $H^*(A, k)$  maps to the graded center of  $H^*(A, R)$ . Now assume  $H$  satisfies **(hfg\*)** and denote again by  $Z$  a graded central  $A$ -module

subalgebra of  $R$ . Write  $W = (Z^{\text{ev}})^A = \bigoplus_{i \geq 0} W_i$ , where  $R$  is finitely generated as a module over  $W$ . Then  $H^*(A, R)$  is a finitely generated module over the noetherian algebra  $H^*(A, W)$  by **(hfg\*)** and hence itself is finitely generated noetherian. We use Corollary 4.4 to conclude that  $H^*(A, W)$  maps to the graded center of  $H^*(A, R)$  since each homogenous component  $W_i$  of  $W$  is a direct sum of copies of the trivial module  $k$ .  $\square$

Similarly, we obtain the following result, where part (i) is a special case of [46, Theorem 1.1].

**Proposition 4.7.** *Let  $A$  be a finite dimensional algebra,  $M$  be a finite dimensional  $A$ -module, and  $R$  be a finite dimensional unital algebra in  $\text{Mod}(A^e)$ . Then*

- (i)  $\text{HH}^*(A)$  maps to the graded center of  $\text{Ext}_A^*(M, M)$ . Moreover, if  $A$  satisfies **(fg)**, then  $\text{Ext}_A^*(M, M)$  is noetherian and a finite module over its graded center.
- (ii)  $\text{HH}^*(A)$  maps to the graded center of  $\text{HH}^*(A, R)$ . Moreover, if  $A$  satisfies **(fg)**, then  $\text{HH}^*(A, R)$  is noetherian and a finite module over its graded center.

*Proof.* Again, apply Theorem 4.3 where  $\mathcal{C}$  is the left derived category of  $\text{Mod}(A^e)$  with  $\otimes = \otimes_A$ ,  $e = A$  and  $T = [1]$  is the shift functor.

(i) Note that  $\text{Ext}_A^*(M, M) \cong \text{HH}^*(A, \text{Hom}_k(M, M))$  and the standard actions of  $\text{HH}^*(A)$  on either one correspond under this isomorphism. By Theorem 4.3,  $\text{HH}^*(A)$  maps to the graded center of  $\text{HH}^*(A, \text{Hom}_k(M, M))$ . Moreover, if  $A$  satisfies **(fg)**, then  $\text{HH}^*(A, \text{Hom}_k(M, M))$  is finitely generated as a module over the noetherian graded central subalgebra  $\text{HH}^*(A)$ , and as a consequence it is noetherian.

(ii) Since  $R$  is  $A$ -unital, by Corollary 4.4,  $\text{HH}^*(A)$  maps to the graded center of  $\text{HH}^*(A, R)$ . Moreover, if  $A$  satisfies **(fg)**,  $\text{HH}^*(A, R)$  is finitely generated as a module over the noetherian graded central subalgebra  $\text{HH}^*(A)$ , and as a consequence it is noetherian.  $\square$

**Lemma 4.8.** *Let  $R$  be a commutative noetherian ring with characteristic  $m > 0$ . Let  $\{E_r^{i,j}\}_{r,i,j}$  be a convergent multiplicative spectral sequence of  $R$ -algebras concentrated in the half plane  $i + j \geq 0$ . Assume that for some  $r_0 \geq 1$ ,*

- (i)  $E_{r_0}^{*,*}$  is a finitely generated module over its graded center (grading by total degree), and
- (ii)  $E_{r_0}^{*,*}$  is a noetherian  $R$ -algebra.
- (iii) The spectral sequence  $\{E_r^{i,j}\}_{r,i,j}$  collapses at some page  $r_1 \geq r_0$ .

Then  $E_\infty^{*,*}$  is a noetherian  $R$ -algebra.

Additionally, let  $\{\tilde{E}_r^{i,j}\}_{r,i,j}$  be a convergent spectral sequence that is a differential bigraded module over  $\{E_r^{i,j}\}_{r,i,j}$ . Suppose that for the same value  $r_0$ ,  $\tilde{E}_{r_0}^{*,*}$  is finitely generated over  $E_{r_0}^{*,*}$ . Then  $\tilde{E}_\infty^{*,*}$  is finitely generated over  $E_\infty^{*,*}$ .

*Proof.* For each  $r \geq 0$ , let  $C_r^{*,*}$  be the graded center of  $E_r^{*,*}$ . Then for each  $c \in C_r^{i,j}$  and  $x \in E_r^{i',j'}$ ,

$$(4.2) \quad cx = (-1)^{(i+j)(i'+j')}xc$$

and  $d(c) \in C_r^{i-1,j} \oplus C_r^{i,j-1}$ . As a consequence of equation (4.2),  $d(c)$  commutes with  $c$ , i.e.,  $cd(c) = d(c)c$ . Thus if  $i$  is odd, then

$$d(c^2) = d(c)c + (-1)^i cd(c) = 0,$$

while if  $i$  is even, since  $m = \text{char}(R)$ , we have

$$d(c^m) = d(c)c^{m-1} + cd(c)c^{m-2} + \dots + c^{m-1}d(c) = md(c)c^{m-1} = 0.$$

It follows that for each  $c$  in  $C_r^{i,j}$  there is a finite positive power of  $c$  that is a cycle (i.e.,  $c^2$  if  $i$  is odd, and  $c^m$  if  $i$  is even).

By hypothesis,  $E_{r_0}^{*,*}$  is a noetherian  $R$ -algebra that is a finitely generated module over  $C_{r_0}^{*,*}$ . By Proposition 2.4(3),  $C_{r_0}^{*,*}$  is noetherian and finitely generated. Let  $c_1, \dots, c_n$  be a set of homogeneous generators of  $C_{r_0}^{*,*}$ . As above, for each  $i$ , some positive power of  $c_i$  is a cycle. By repeating the process, for each  $i$ , there is a finite positive power, say  $c_i^{t_i}$ , that is a permanent cycle since the spectral sequence  $\{E_r^{i,j}\}_{r,i,j}$  collapses at some page  $r_1 \geq r_0$ . Let

$$A_{r_0}^{*,*} := R\langle c_1^{t_1}, \dots, c_n^{t_n} \rangle \subseteq C_{r_0}^{*,*},$$

that is,  $A_{r_0}^{*,*}$  is the subalgebra of  $C_{r_0}^{*,*}$  generated by  $c_1^{t_1}, \dots, c_n^{t_n}$ . By its definition,  $A_{r_0}^{*,*}$  consists of permanent cycles and  $C_{r_0}^{*,*}$  is finitely generated as a module over  $A_{r_0}^{*,*}$ . It follows that  $E_{r_0}^{*,*}$  is also a finitely generated module over  $A_{r_0}^{*,*}$ . For all  $r > r_0$ , let

$$A_r^{*,*} := A_{r-1}^{*,*}/B_{r-1}^{*,*} \subseteq E_r^{*,*},$$

the subalgebra of  $E_r^{*,*}$  given by the quotient of  $A_{r-1}^{*,*}$  by its ideal  $B_{r-1}^{*,*}$  consisting of coboundaries. By construction, we have a sequence of algebras:

$$A_{r_0}^{*,*} \twoheadrightarrow \dots \twoheadrightarrow A_r^{*,*} \twoheadrightarrow A_{r+1}^{*,*} \twoheadrightarrow \dots,$$

for which each  $A_i^{*,*}$  is a subalgebra of  $E_i^{*,*}$ . For each  $r \geq r_0$ , let  $\Lambda_r^{*,*}$  be the subalgebra consisting of *all permanent cycles* in  $E_r^{*,*}$ . A calculation shows that since  $A_r^{*,*}$  consists of permanent cycles,  $d_r(E_r^{*,*})$  is an  $A_r^{*,*}$ -submodule of  $\Lambda_r^{*,*}$ .

Writing  $\Lambda_{r+1}^{*,*} := \Lambda_r^{*,*}/d_r(E_r^{*,*})$  for each  $r$ , we have a sequence of  $A_{r_0}^{*,*}$ -modules:

$$\Lambda_{r_0}^{*,*} \twoheadrightarrow \dots \twoheadrightarrow \Lambda_r^{*,*} \twoheadrightarrow \Lambda_{r+1}^{*,*} \twoheadrightarrow \dots$$

Now  $\Lambda_{r_0}^{*,*}$  is an  $A_{r_0}^{*,*}$ -submodule of  $E_{r_0}^{*,*}$ , and so is a noetherian  $A_{r_0}^{*,*}$ -module. Let  $K_r$  be the kernel of the surjection from  $\Lambda_{r_0}^{*,*}$  to  $\Lambda_r^{*,*}$ . Then  $K_r$  is also a noetherian  $A_{r_0}^{*,*}$ -module. There is an increasing chain of submodules of  $\Lambda_{r_0}^{*,*}$ :

$$K_{r_0} \subseteq K_{r_0+1} \subseteq \dots$$

Since  $\Lambda_{r_0}^{*,*}$  is noetherian, this chain stabilizes, that is  $K_s = K_{s+1} = \dots$ , for some  $s \geq r_0$ . Therefore,  $\Lambda_s^{*,*} = E_\infty^{*,*}$ , and  $E_\infty^{*,*}$  is itself a noetherian  $A_{r_0}^{*,*}$ -module.

By [12, Proposition 2.1],  $E_\infty^{*,*}$  is a noetherian module over  $\text{Tot}(A_{r_0}^{*,*})$ . Since  $A_{r_0}^{*,*}$  is finitely generated, it follows that  $E_\infty^{*,*}$  is a noetherian  $R$ -algebra.

For the remaining statement regarding  $\tilde{E}^{i,j}$ , under the above set up and hypotheses,  $\tilde{E}_{r_0}^{*,*}$  is also a finitely generated module over  $A_{r_0}^{*,*}$ . For each  $r \geq r_0$ , let  $\tilde{\Lambda}_r^{*,*}$  be the subalgebra consisting of *all permanent cycles* in  $\tilde{E}_r^{*,*}$  and define  $\tilde{\Lambda}_{r+1}^{*,*} := \tilde{\Lambda}_r^{*,*}/d_r(\tilde{E}_r^{*,*})$ . Now  $\tilde{\Lambda}_{r_0}^{*,*}$  is an  $A_{r_0}^{*,*}$ -submodule of  $\tilde{E}_{r_0}^{*,*}$ , and so is a finitely generated  $A_{r_0}^{*,*}$ -module. Similar arguments show that  $\tilde{E}_\infty^{*,*}$  is a noetherian  $A_{r_0}^{*,*}$ -module and is finitely generated over  $E_\infty^{*,*}$ .  $\square$

**(Hochschild) Cohomology of filtered algebras:** Let  $A$  be a finite dimensional filtered (Hopf) algebra, and denote by  $\text{gr } A$  the corresponding associated graded (Hopf) algebra. Then there exists a May spectral sequence computing the (Hochschild) cohomology of  $A$  in terms of the (Hochschild) cohomology of  $\text{gr } A$  as its first page. Similarly, there is a May spectral sequence that computes the (Hochschild) cohomology of  $A$  with coefficients in any finite  $A$ -module  $M$ . Moreover, these spectral sequences inherit structures that are compatible with the cup product in the (Hochschild) cohomology of  $A$  and its module structure on that of  $M$  (see Appendix A.1 and Appendix A.2).

**Theorem 4.9.** *Let  $A$  be a finite dimensional filtered algebra (resp. Hopf algebra) over a field  $k$  of positive characteristic. If the associated graded algebra (resp. Hopf algebra)  $\text{gr } A$  satisfies **(fg)***

(resp. **(hfg)**) and the May spectral sequence used to compute  $\mathrm{HH}^*(A)$  (resp.  $\mathrm{H}^*(A, k)$ ) collapses at certain page, then  $A$  satisfies **(fg)** (resp. **(hfg)**).

*Proof.* It follows from Lemma 4.8 and the discussion above since the (Hochschild) cohomology ring of  $A$  is always graded commutative.  $\square$

**Cohomology of crossed products:** Let  $A = R\#_\sigma H$  be the crossed product of two finite dimensional Hopf algebras  $R$  and  $H$  over a field  $k$  with respect to a cocycle  $\sigma$  (see [33]). We assume that the augmentations of  $R$  and  $H$  are preserved under the crossed product so that  $A$  is again augmented. In this case, there are Lyndon-Hochschild-Serre spectral sequences associated to the crossed product  $A = R\#_\sigma H$  and any finite dimensional  $A$ -module  $M$ :

$$\begin{aligned} E_2^{p,q}(A) = \mathrm{H}^p(H, \mathrm{H}^q(R, k)) &\implies \mathrm{H}^{p+q}(A, k) = E_\infty^{p,q}(A), \\ E_2^{p,q}(M) = \mathrm{H}^p(H, \mathrm{H}^q(R, M)) &\implies \mathrm{H}^{p+q}(A, M) = E_\infty^{p,q}(M). \end{aligned}$$

These spectral sequences inherit structures that are compatible with the multiplicative structure of  $\mathrm{H}^*(A, k)$  and its module structure on  $\mathrm{H}^*(A, M)$  (see Appendix A.3).

**Theorem 4.10.** *Retain the notations above. Further assume that  $R$  satisfies **(hfg)** and  $H$  satisfies **(hfg\*)**. Then  $A = R\#_\sigma H$  satisfies **(hfg)** if the corresponding LHS spectral sequence  $E_r^{*,*}(A)$  collapses for*

- (i)  $r = 2$  when  $\mathrm{char}(k) = 0$ , or
- (ii) some  $r \geq 2$  when  $\mathrm{char}(k) > 0$ .

*Proof.* Since  $R$  satisfies **(hfg)**, the cohomology ring  $\mathrm{H}^*(R, k)$  is graded commutative and finitely generated. In view of Proposition A.4 and the fact that  $H$  satisfies **(hfg\*)**, we know  $E_2^{*,*}(A) = \mathrm{H}^*(H, \mathrm{H}^*(R, k))$  is a finitely generated and noetherian algebra over  $E_2^{0,0}(A) = k$  and it is a finite module over its graded center. Moreover since  $\mathrm{H}^*(R, M)$  is finite over  $\mathrm{H}^*(R, k)$ , Lemma 3.3(iv) implies that  $E_2^{*,*}(M) = \mathrm{H}^*(H, \mathrm{H}^*(R, M))$  is a finite module over  $E_2^{*,*}(A) = \mathrm{H}^*(H, \mathrm{H}^*(R, k))$ . Thus the result holds in characteristic zero and for positive characteristic we can apply Lemma 4.8 since all the assumptions there are satisfied when  $r_0 = 2$ .  $\square$

**Hochschild cohomology of smash products:** Now let  $A = R\#H$  be the smash product of a finite dimensional Hopf algebra  $H$  and a finite dimensional  $H$ -module algebra  $R$ . There are Lyndon-Hochschild-Serre spectral sequences associated to the smash product  $A = R\#H$  and any finite bimodule  $M$  over  $A$ :

$$\begin{aligned} E_2^{p,q}(A) = \mathrm{H}^p(H, \mathrm{HH}^q(R, A)) &\implies \mathrm{HH}^{p+q}(A) = E_\infty^{p,q}(A), \\ E_2^{p,q}(M) = \mathrm{H}^p(H, \mathrm{HH}^q(R, M)) &\implies \mathrm{HH}^{p+q}(A, M) = E_\infty^{p,q}(M). \end{aligned}$$

These spectral sequences naturally inherit structures that are compatible with the multiplicative structure of  $\mathrm{HH}^*(A)$  and its module structure on  $\mathrm{HH}^*(A, M)$  (see Appendix A.4).

**Theorem 4.11.** *Retain the notations above. Further assume that  $R$  satisfies **(fg)** and  $H$  is cocommutative. Then  $A = R\#H$  satisfies **(fg)** if the corresponding LHS spectral sequence  $E_r^{*,*}(A)$  collapses for*

- (i)  $r = 2$  when  $\mathrm{char}(k) = 0$ , or
- (ii) some  $r \geq 2$  when  $\mathrm{char}(k) > 0$ .

*Proof.* Since  $R$  satisfies **(fg)**,  $\mathrm{HH}^*(R, A)$  is a finite module over  $\mathrm{HH}^*(R)$ . This implies that  $\mathrm{HH}^*(R, A)$  is finitely generated and noetherian since  $\mathrm{HH}^*(R)$  is. Furthermore,  $\mathrm{HH}^*(R)$  is mapped to the graded

center of  $\mathrm{HH}^*(R, A)$  via the embedding  $R \hookrightarrow A$  by Proposition 4.7(ii). Next we show that the image of  $\mathrm{HH}^*(R)$  is an  $H$ -module subalgebra of  $\mathrm{HH}^*(R, A)$ .

Using the notation in Appendix A.4, let  $K_\bullet$  be an  $H$ -equivariant  $R$ -bimodule resolution of  $R$ . Then  $\mathrm{HH}^*(R, A) \cong \mathrm{H}^*(\mathrm{Hom}_{R^e}(K_\bullet, A))$  and  $\mathrm{HH}^*(R) \cong \mathrm{H}^*(\mathrm{Hom}_{R^e}(K_\bullet, R))$ . Choose any  $f \in \mathrm{Hom}_{R^e}(K_\bullet, R) \subset \mathrm{Hom}_{R^e}(K_\bullet, A)$ ,  $h \in H$ , and  $u \in K_\bullet$ . Applying the  $H$ -action given in Appendix A.4 and the assumption that  $H$  is cocommutative, we find that

$$\begin{aligned} (f \cdot h)(u) &= \sum S(h_1)f(h_2 \cdot u)h_3 = \sum [S(h_2) \cdot f(h_3 \cdot u)]S(h_1)h_4 \\ &= \sum [S(h_3) \cdot f(h_4 \cdot u)]S(h_1)h_2 = \sum S(h_1) \cdot f(h_2 \cdot u) \in B. \end{aligned}$$

Hence,  $\mathrm{Hom}_{R^e}(K_\bullet, R)$  is an  $H$ -invariant subcomplex of  $\mathrm{Hom}_{R^e}(K_\bullet, A)$ . Passing to homology, the image of  $\mathrm{HH}^*(R)$  is an  $H$ -module subalgebra of  $\mathrm{HH}^*(R, A)$ .

As a result of Friedlander and Suslin [17],  $H$  satisfies **(hfg)** and hence it satisfies **(hfg\*)** by Lemma 3.3 since  $H$  is cocommutative (see Proposition 2.5). By Proposition 4.6(ii), we know  $E_2^{*,*}(A) = \mathrm{H}^*(H, \mathrm{HH}^*(R, A))$  is a finitely generated noetherian algebra over the center  $E_2^{0,0}(A) = Z(A)$  of  $A$  and it is a finite module over its graded center. Moreover since  $\mathrm{HH}^*(R, M)$  is a finite module over  $\mathrm{HH}^*(R, A)$ , Lemma 3.3(iv) implies that  $E_2^{*,*}(M) = \mathrm{H}^*(H, \mathrm{HH}^*(R, M))$  is finitely generated over  $E_2^{*,*}(A) = \mathrm{H}^*(H, \mathrm{HH}^*(R, A))$ . Thus, the result holds in characteristic zero. For positive characteristic, we can apply Lemma 4.8 since all the assumptions there are satisfied when  $r_0 = 2$ .  $\square$

## 5. A LIFTING METHOD FOR THE FINITE GENERATION CONDITIONS VIA REDUCTION MODULO $p$

In this section, we use the reduction modulo  $p$  method in number theory to deal with the finite generation conditions for the (Hochschild) cohomology ring of (Hopf) algebras over a field of characteristic zero, or mainly over the field of complex numbers  $\mathbb{C}$ .

Our first result suggests that regarding the finite generation conditions over a field of characteristic zero, it suffices to work over  $\mathbb{C}$ .

**Lemma 5.1.** *Let  $k$  be an arbitrary field of characteristic zero, and  $A$  be a finite dimensional algebra (resp. Hopf algebra) over  $k$ . Then there is some finite dimensional algebra  $A'$  (resp. Hopf algebra) over  $\mathbb{C}$  such that  $A$  satisfies **(fg)** (resp. **(hfg)**) if and only if  $A'$  satisfies **(fg)** (resp. **(hfg)**).*

*Proof.* Here we only treat the case when  $A$  is a finite dimensional algebra over  $k$  with condition **(fg)**. The argument for  $A$  being a Hopf algebra with condition **(hfg)** is similar. Since  $k$  is of characteristic zero, the prime field of  $k$  is  $\mathbb{Q}$ . Fix a finite basis  $x_1, \dots, x_n$  of  $A$ . We can write the multiplication in  $A$  as  $x_i x_j = \sum_\ell \alpha_{ij}^\ell x_\ell$  for some  $\alpha_{ij}^\ell \in k$ . Take  $K = \mathbb{Q}(\alpha_{ij}^\ell)$  to be the subfield of  $k$  by joining all the coefficients  $\alpha_{ij}^\ell$  to its prime field  $\mathbb{Q}$ . Then one can define another algebra  $B$  over  $K$  with the bases  $x_i$ 's and the same multiplication rule  $x_i x_j = \sum_\ell \alpha_{ij}^\ell x_\ell$ . It is clear that  $B \otimes_K k \cong A$ . Now since  $\mathbb{C}$  is algebraically closed over  $\mathbb{Q}$  and has infinitely many transcendental numbers, one can embed  $K$  into  $\mathbb{C}$ . Let  $A' := B \otimes_K \mathbb{C}$  which is a finite dimensional complex algebra. By Lemma 3.1 and Remark 3.2,  $A$  satisfies **(fg)**  $\Leftrightarrow B$  satisfies **(fg)**  $\Leftrightarrow A'$  satisfies **(fg)**.  $\square$

**Lemma 5.2.** *Let  $R = \bigoplus_{i \geq 0} R_i$  be a connected graded commutative algebra over a base field  $K \subset \mathbb{C}$ . If  $R \otimes_K k$  is noetherian for some field extension  $k/K$ , then  $R \otimes_K \mathbb{C}$  is noetherian.*

*Moreover, if  $M = \bigoplus_{i \geq 0} M_i$  is a graded module over  $R$  such that  $M \otimes_K k$  is finite over  $R \otimes_K k$ , then  $M \otimes_K \mathbb{C}$  is finite over  $R \otimes_K \mathbb{C}$ .*

*Proof.* By Proposition 2.4(ii), we know  $R \otimes_K k$  is finitely generated over  $k$ , say by homogenous elements  $f_1, \dots, f_r$ . Choose a  $K$ -basis  $\{x_i\}$  for  $R$ . Denote by  $F$  the subfield of  $k$  by joining all the coefficients appearing in  $f_1, \dots, f_r$  as  $k$ -linear combinations of the basis  $\{x_i\}$  to  $K$ . Then  $f_1, \dots, f_r$  belong to  $R \otimes_K F$  and they generate a  $F$ -subalgebra in  $R \otimes_K F$ , which is denoted by  $T$ . Note that  $R \otimes_K k$  is locally finite and has a Hilbert series. It is clear that  $T \otimes_F k \cong R \otimes_K k$ . Since base field extension does not change the Hilbert series,  $T \subseteq R \otimes_K F \subseteq R \otimes_K k$  share the same Hilbert series. This implies that  $T = R \otimes_K F$  and  $R \otimes_K F$  is finitely generated. Now by the fact that  $\mathbb{C}$  is algebraically closed over  $K$  and has uncountably many transcendental numbers, we can embed  $F$  into  $\mathbb{C}$ . Hence  $(R \otimes_K F) \otimes_F \mathbb{C} \cong R \otimes_K \mathbb{C}$ . So  $R \otimes_K \mathbb{C}$  is finitely generated and hence is noetherian by Proposition 2.4(ii).

Finally for any graded module  $M$  over  $R$ , take finitely many homogenous generators  $f_1, \dots, f_r$  of  $M \otimes_K k$  over  $R \otimes_K k$ . Then there is a middle field  $K \subseteq F \subseteq k$  such that  $f_1, \dots, f_r$  belong to  $M \otimes_K F$ . By a similar argument to that above, we can consider  $T$  as the submodule generated by  $f_1, \dots, f_r$  in  $M \otimes_K F$  and conclude that  $T \otimes_F \mathbb{C} \cong M \otimes_K \mathbb{C}$  is finitely generated over  $R \otimes_K \mathbb{C}$  via some embedding of  $F$  into  $\mathbb{C}$ .  $\square$

**Lemma 5.3.** *Let  $R = \bigoplus_{i \geq 0} R_i$  be a graded commutative algebra over a noetherian commutative base ring  $R_0$  that is locally finite over  $R_0$ , and  $I$  be any ideal of  $R_0$ . If  $R \otimes_{R_0} (R_0/I)$  is noetherian, then  $R \otimes_{R_0} (\varprojlim_i R_0/I^i)$  is noetherian.*

*Moreover, let  $M = \bigoplus_{i \geq 0} M_i$  be a graded module over  $R$  that is locally finite over  $R_0$ . If  $M \otimes_{R_0} (R_0/I)$  is finite over  $R \otimes_{R_0} (R_0/I)$ , then  $M \otimes_{R_0} (\varprojlim_i R_0/I^i)$  is finite over  $R \otimes_{R_0} (\varprojlim_i R_0/I^i)$ .*

*Proof.* We write  $\widehat{R}_0 = \varprojlim_i R_0/I^i$  and  $\widehat{R} = R \otimes_{R_0} \widehat{R}_0$ . For any  $R_0$ -module  $M$ , we denote the natural map

$$\varphi_N : M \otimes_{R_0} \widehat{R}_0 \longrightarrow \varprojlim_i M/I^i M.$$

Suppose  $M$  is finite over  $R_0$ . By taking a finite presentation  $R_0^m \rightarrow R_0^n \rightarrow M \rightarrow 0$  of  $M$ , we get the following commutative diagram:

$$\begin{array}{ccccccc} \widehat{R}_0^m & \longrightarrow & \widehat{R}_0^n & \longrightarrow & M \otimes_{R_0} \widehat{R}_0 & \longrightarrow & 0 \\ \downarrow \varphi_{R_0^m} & & \downarrow \varphi_{R_0^n} & & \downarrow \varphi_M & & \\ \varprojlim_i (R_0/I^i)^m & \longrightarrow & \varprojlim_i (R_0/I^i)^n & \longrightarrow & \varprojlim_i M/I^i M & \longrightarrow & 0 \end{array}$$

The first row above is exact since it is obtained by applying  $- \otimes_{R_0} \widehat{R}_0$  to the finite presentation of  $M$ . Moreover, we know  $\varprojlim_i^1 R_0/I^i = 0$  for the maps in the inverse sequence

$$R_0/I \longleftarrow R_0/I^2 \longleftarrow R_0/I^3 \longleftarrow \dots$$

satisfies the Mittag-Leffler condition. Then by taking the inverse limit of the exact sequence  $(R_0/I^i)^r \rightarrow (R_0/I^i)^s \rightarrow M/I^i M \rightarrow 0$  over  $i \geq 1$ , we get the second row above is exact. Now since  $\varphi_{R_0^m}$  and  $\varphi_{R_0^n}$  are isomorphisms, we know  $\varphi_M$  is an isomorphism whenever  $M$  is finite over  $R_0$ . As a

consequence since  $R$  is locally finite over  $R_0$ , we have

$$\widehat{R} = \left( \bigoplus_{j \geq 0} R_j \right) \otimes_{R_0} \widehat{R}_0 \cong \bigoplus_{j \geq 0} \left( R_j \otimes_{R_0} \widehat{R}_0 \right) \cong \bigoplus_{j \geq 0} \left( \varprojlim_i R_j / I^i R_j \right).$$

So  $\widehat{R}$  has an  $I$ -adic filtration such that

$$\begin{aligned} \mathrm{gr}_I \widehat{R} &= \bigoplus_{i \geq 0} \left( I^i \widehat{R} / I^{i+1} \widehat{R} \right) \cong \bigoplus_{j \geq 0} \left( \bigoplus_{i \geq 0} I^i R_j / I^{i+1} R_j \right) \cong \bigoplus_{j \geq 0} \left( \bigoplus_{i \geq 0} R_j \otimes_{R_0} I^i / I^{i+1} \right) \\ &\cong R \otimes_{R_0} \left( \bigoplus_{i \geq 0} I^i / I^{i+1} \right) \cong R \otimes_{R_0} \mathrm{gr}_I \widehat{R}_0. \end{aligned}$$

Since  $R_0$  is noetherian,  $I$  is finitely generated, say  $I = (a_1, \dots, a_n)$ . Thus there is a surjection  $(R_0/I)[t_1, \dots, t_n] \twoheadrightarrow \mathrm{gr}_I \widehat{R}_0$  given by  $t_i \mapsto a_i$  in  $I/I^2$ . Then we get a surjection  $(R \otimes_{R_0} R_0/I)[t_1, \dots, t_n] \twoheadrightarrow \mathrm{gr}_I \widehat{R}$ . By assumption we know  $R \otimes_{R_0} (R_0/I)$  is noetherian. Hence  $\mathrm{gr} \widehat{R}$  is noetherian and so is  $\widehat{R}$ . The statement for finite generation of modules can be proved similarly.  $\square$

**Definition 5.4.** Let  $A$  be a finite dimensional Hopf algebra over  $\mathbb{C}$ . We say  $A$  can be defined over some algebraic number field  $K/\mathbb{Q}$  if there is some Hopf  $K$ -subalgebra  $B$  of  $A$  such that  $A \cong B \otimes_K \mathbb{C}$ .

**Theorem 5.5.** *Let  $A$  be a finite-dimensional complex Hopf algebra that can be defined over some algebraic number field  $K$ . Then there exists a finite dimensional Hopf algebra  $A'$  over some finite field  $F$  such that if  $A'$  satisfies **(hfg)** then so does  $A$ .*

*Moreover, there exists some localization  $\mathcal{R}$  of  $\mathcal{O}_K$  finitely generated over  $\mathbb{Z}$  and some free Hopf  $R$ -subalgebra  $B$  of  $A$  such that  $A \cong B \otimes_R \mathbb{C}$ , where we can let  $A' = B \otimes_{\mathcal{R}} F$  with  $F = \mathcal{R}/(p)$  for some prime  $p$ .*

*Proof.* Since  $A$  can be defined over  $K$ , there exists some Hopf  $K$ -subalgebra, denoted by  $L$ , such that  $L \otimes_K \mathbb{C} \cong A$ . In view of Lemma 3.1, by replacing  $K$  with a possible finite field extension, we may assume the Jacobson radical  $J_L$  of  $L$  splits in  $L$ . Thus we can choose a finite basis  $x_1, \dots, x_n$  for  $L$  such that  $J_L = \mathrm{span}_K(x_1, \dots, x_m)$  and  $J_L \otimes_K \mathbb{C} \cong J_A$  the Jacobson radical of  $A$ . Denote by  $\mathcal{S}$  the subset of  $K$  which consists of all the coefficients when we apply (co)multiplication, (co)unit, and antipode of  $L$  to the above fixed basis. Further denote by  $\mathcal{R}$  the localization of  $\mathcal{O}_K$  at the multiplicative set generated by all the denominators appeared in  $\mathcal{S}$ . Since  $\dim_K L < \infty$ ,  $\mathcal{S}$  is finite and  $\mathcal{R}$  is a finitely generated  $\mathbb{Z}$ -algebra. As a consequence, there is some prime  $p$  such that  $F = \mathcal{R}/(p)$  is a finite field. Set  $B := \mathrm{span}_{\mathcal{R}}(x_1, \dots, x_n)$ . Since  $\mathcal{S} \subset \mathcal{R}$ , one can check that  $B$  is a free Hopf  $\mathcal{R}$ -subalgebra of  $A$  satisfying

$$B \otimes_{\mathcal{R}} \mathbb{C} \cong (B \otimes_{\mathcal{R}} K) \otimes_K \mathbb{C} \cong L \otimes_K \mathbb{C} \cong A.$$

Moreover,  $W := \mathrm{span}_{\mathcal{R}}(x_1, \dots, x_m)$  is a finite  $B$ -module satisfying

$$W \otimes_{\mathcal{R}} \mathbb{C} \cong (W \otimes_{\mathcal{R}} K) \otimes_K \mathbb{C} \cong J_L \otimes_K \mathbb{C} \cong J_A.$$

We use the language of differential graded algebras and modules to describe the cohomology ring  $H^*(B, \mathcal{R})$  of  $B$  and the cohomology  $H^*(B, W)$  of  $B$  with coefficients in  $W$ . We apply  $- \otimes_B \mathcal{R}$  to the bar resolution of  $B$  over  $\mathcal{R}$ ,

$$\cdots \xrightarrow{\partial_3} B \otimes_{\mathcal{R}} B \otimes_{\mathcal{R}} B \otimes_{\mathcal{R}} B \xrightarrow{\partial_2} B \otimes_{\mathcal{R}} B \otimes_{\mathcal{R}} B \xrightarrow{\partial_1} B \otimes_{\mathcal{R}} B \xrightarrow{\mu} B,$$

where  $\partial_i(b_0 \otimes \cdots \otimes b_{i+1}) = \sum_{j=0}^i (-1)^j b_0 \otimes \cdots \otimes b_j b_{j+1} \otimes \cdots \otimes b_{i+1}$ . Since  $B$  is free over  $\mathcal{R}$ , we get a projective resolution of  $R$  in the category of left  $B$ -modules. Therefore, we obtain two complexes by applying  $\text{Hom}_B(-, \mathcal{R})$  and  $\text{Hom}_B(-, W)$  to this resolution, namely

$$(5.1) \quad C^n(B, \mathcal{R}) := \text{Hom}_{\mathcal{R}}(B^{\otimes n}, \mathcal{R}) \text{ and } C^n(B, W) := \text{Hom}_{\mathcal{R}}(B^{\otimes n}, W),$$

where we omit the formulas of the corresponding differentials. Note that  $C^*(B, \mathcal{R})$  is a differential graded algebra with the cup product given by, for any  $f \in C^m(B, \mathcal{R})$  and  $g \in C^n(B, \mathcal{R})$ ,

$$(5.2) \quad f \smile g(b_1 \otimes \cdots \otimes b_{m+n}) = f(b_1 \otimes \cdots \otimes b_m)g(b_{m+1} \otimes \cdots \otimes b_{m+n}).$$

Moreover,  $C^\bullet(B, W)$  is differential graded module over  $C^\bullet(B, \mathcal{R})$  with the module action similar to (5.2) by considering  $g \in C^n(B, W)$ . After taking the cohomology of the two complexes (5.1), we obtain  $H^*(B, \mathcal{R})$  and  $H^*(B, W)$ , where the Yoneda product is compatible with the cup product. Since  $B$  is a free module over  $\mathcal{R}$  of finite rank, it is projective and so is  $B^{\otimes n}$ . Hence we have

$$\begin{aligned} C^n(B, \mathcal{R}) \otimes_{\mathcal{R}} F &= \text{Hom}_{\mathcal{R}}(B^{\otimes n}, \mathcal{R}) \otimes_{\mathcal{R}} F \cong \text{Hom}_F((B^{\otimes n}) \otimes_{\mathcal{R}} F, F) \\ &\cong \text{Hom}_F((B \otimes_{\mathcal{R}} F)^{\otimes n}, F) = \text{Hom}_F(A'^{\otimes n}, F) =: C^n(A', F), \end{aligned}$$

which calculates the cohomology ring  $H^*(A', F)$ . Likewise,  $C^n(B, W) \otimes_{\mathcal{R}} F = \text{Hom}_F(A'^{\otimes n}, W') =: C^n(A', W')$  with  $W' = W \otimes_{\mathcal{R}} F$  a finite module over  $A'$ . By the hypothesis that  $A'$  satisfies **(hfg)**, we know  $H^*(C^\bullet(A', W'))$  is noetherian over  $H^*(C^\bullet(A', F))$ .

Now as a localization of  $\mathcal{O}_K$ ,  $\mathcal{R}$  is a Dedekind domain and hence is hereditary. So we have a project resolution  $0 \rightarrow (p) \rightarrow \mathcal{R} \rightarrow F \rightarrow 0$  in the category of left  $\mathcal{R}$ -modules. After tensoring the resolution with the complex  $C^\bullet(B, \mathcal{R})$ , we get a double complex which yields a spectral sequence

$$E_2^{pq} := \text{Tor}_q^{\mathcal{R}}(H^p(C^\bullet(B, \mathcal{R}), F)) \Rightarrow E_\infty^{pq} := H^{p+q}(C^\bullet(B, \mathcal{R}) \otimes_{\mathcal{R}} F) \cong H^{p+q}(A', F),$$

which collapses at the  $E_2$  page.

Next note that the double complex  $C^\bullet(B, \mathcal{R})$  has a multiplicative structure that induces a multiplicative structure on the spectral sequence. Deduced by the filtration on the double complex, we get that  $E_2^{p(q \geq 1)} = \text{Tor}_{\geq 1}^{\mathcal{R}}(H^p(C^\bullet(B, \mathcal{R}), F))$  is an ideal in page  $E_2$ . That is,  $E_2^{p0} = H^p(C^\bullet(B, \mathcal{R})) \otimes_{\mathcal{R}} F$  is a quotient of the  $E_2 = E_\infty$  page. Then  $H^*(C^\bullet(B, \mathcal{R})) \otimes_{\mathcal{R}} F$  is a quotient of the noetherian ring  $H^*(A', F)$ , and as a consequence is noetherian itself. By a similar argument, one can show that  $H^*(C^\bullet(B, W)) \otimes_{\mathcal{R}} F$  is a finite module over  $H^*(C^\bullet(B, \mathcal{R})) \otimes_{\mathcal{R}} F$ . By Lemma 5.3,

$$H^*(C^\bullet(B, \mathcal{R})) \otimes_{\mathcal{R}} \left( \varprojlim_i \mathcal{R}/(p^i) \right) = H^*(C^\bullet(B, \mathcal{R})) \otimes_{\mathcal{R}} \widehat{\mathcal{R}}$$

is noetherian. Note that  $H^*(C^\bullet(B, \mathcal{R})) \otimes_{\mathcal{R}} \widehat{\mathcal{R}}$  is graded commutative and finitely generated. So  $H^*(C^\bullet(B, \mathcal{R})) \otimes_{\mathcal{R}} \text{Frac} \widehat{\mathcal{R}}$  is noetherian and finitely generated. Similarly, one argues that, since  $H^*(C^\bullet(B, W)) \otimes_{\mathcal{R}} \mathcal{R}/(p)$  is finite over  $H^*(C^\bullet(B, \mathcal{R})) \otimes_{\mathcal{R}} \mathcal{R}/(p)$ , one gets that  $H^*(C^\bullet(B, W)) \otimes_{\mathcal{R}} \widehat{\mathcal{R}}$  is finite over  $H^*(C^\bullet(B, \mathcal{R})) \otimes_{\mathcal{R}} \widehat{\mathcal{R}}$ . As a consequence,  $H^*(C^\bullet(B, W)) \otimes_{\mathcal{R}} \text{Frac} \widehat{\mathcal{R}}$  is finite over  $H^*(C^\bullet(B, \mathcal{R})) \otimes_{\mathcal{R}} \text{Frac} \widehat{\mathcal{R}}$ .

Note that  $\text{Frac} \mathcal{R} = \text{Frac} \mathcal{O}_K = K$  and write  $\text{Frac} \widehat{\mathcal{R}} = k$ . Let  $T := H^*(C^\bullet(B, \mathcal{R})) \otimes_{\mathcal{R}} K$  and  $M := H^*(C^\bullet(B, W)) \otimes_{\mathcal{R}} K$ . By previous discussion, we have  $T \otimes_K k$  is noetherian and  $M \otimes_K k$  is finite over  $T \otimes_K k$ . Note that  $K \subset \mathbb{C}$  is a subfield. By Lemma 5.2, one sees that  $T \otimes_K \mathbb{C}$  is noetherian and  $M \otimes_K \mathbb{C}$  is finite over  $T \otimes_K \mathbb{C}$ . Finally, since  $\mathbb{C}$  is flat over  $K = \text{Frac} \mathcal{R}$  which is also flat over  $\mathcal{R}$ , we have

$$T \otimes_{\mathcal{R}} \mathbb{C} = H^*(C^\bullet(B, \mathcal{R})) \otimes_{\mathcal{R}} \mathbb{C} \cong H^*(C^\bullet(B, \mathcal{R}) \otimes_{\mathcal{R}} \mathbb{C}) \cong H^*(C^\bullet(A, \mathbb{C})) \cong H^*(A, \mathbb{C})$$

and

$$M \otimes_{\mathcal{R}} \mathbb{C} = H^*(C^*(B, W)) \otimes_{\mathcal{R}} \mathbb{C} \cong H^*(C^*(B, W) \otimes_{\mathcal{R}} \mathbb{C}) \cong H^*(C^*(A, J_A)) \cong H^*(A, J_A).$$

Then we conclude that  $A$  satisfies **(hfg)** by Proposition 3.4.  $\square$

The following result can be proved analogously.

**Theorem 5.6.** *Let  $A$  be a finite-dimensional complex algebra that can be defined over some algebraic number field. Then there exists a finite dimensional algebra  $A'$  over some finite field such that if  $A'$  satisfies **(fg)**, then so does  $A$ .*

*Remark 5.7.* Our lifting method can be restricted only for the finite generation of cohomology rings and it also can be extended to finite dimensional augmented algebras in zero characteristic. Moreover, we can always assume the resulting (Hopf or augmented) algebra after the reduction modulo  $p$  is over an algebraically closed field of characteristic  $p$  by a field extension in view of Lemma 3.1 and Remark 3.2.

**Applications.** We provide here some applications for the finite generation conditions when the resulting Hopf algebra via reduction modulo  $p$  is the smash product of a quantum complete intersection with a semisimple Hopf algebra. In particular, our result is applicable for finite dimensional pointed Hopf algebras of diagonal type over the complex numbers.

**Lemma 5.8.** *Let  $R$  be a finite dimensional (resp. augmented) algebra satisfying **(fg)** (resp. **(hfg)**), and  $H$  a semisimple Hopf algebra. Suppose there is an  $H$ -action on  $R$  (resp. preserving the augmentation of  $R$ ). Then the smash product  $R\#H$  satisfies **(fg)** (resp. **(hfg)**).*

*Proof.* Here we prove the augmented case, and the argument for the algebra case is similar. Since  $H$  is semisimple, we know  $H^*(R\#H, k) \cong H^*(R, k)^H$ . By hypothesis,  $H^*(R, k)$  is finitely generated and noetherian, and so is its invariant ring  $H^*(R, k)^H$  by [33, Corollary 4.3.5]. Let  $M$  be a finite  $R\#H$ -module. Since  $M$  is also finite over  $R$  by the restriction, we have  $H^*(R, M)$  is noetherian over  $H^*(R, k)$  by hypothesis. Thus the submodule  $H^*(R\#H, M) \cong H^*(R, M)^H$  of  $H^*(R, M)$  is finite over  $H^*(R, k)$ . Then we can conclude that  $H^*(R\#H, M)$  is finite over  $H^*(R\#H, k)$  as  $H^*(R, k)$  is finite over  $H^*(R, k)^H$  by [33, Theorem 4.4.2].  $\square$

We will work with some **quantum complete intersections**  $R$ , as recalled here.

Let  $t$  be a positive integer and for each  $1 \leq i \leq t$ , let  $N_i \geq 2$  be an integer. Let  $q_{ij} \in k^\times$  for  $1 \leq i < j \leq t$ . Let  $R$  be the  $k$ -algebra generated by  $x_1, \dots, x_t$ , subject to relations

$$x_i x_j = q_{ij} x_j x_i \quad \text{and} \quad x_i^{N_i} = 0,$$

for all  $i < j$  and all  $i$ . It is clear that  $R$  is augmented since  $R$  is local with the unique maximal ideal  $(x_1, \dots, x_t)$ .

The cohomology ring  $H^*(R, k)$  is finitely generated and noetherian. See [7, Theorem 5.3] and [28, Theorem 4.1]. The latter reference corrects some small errors in the relations of the former, but omits the necessary distinction of the cases where  $N_i = 2$  for some  $i$ . (The proof of exactness of the resolution in [28] requires characteristic 0, however it is essentially the same resolution as that given in [7], which is proven to be exact in any characteristic.) So in this case, we know  $R$  satisfies **(hfg)** by Proposition 3.4 since  $R$  is local.

Hochschild cohomology behaves somewhat differently:  $\text{HH}^*(R)$  itself is in fact finite dimensional for some choices of values of  $q_{ij}$ . By [7, Theorem 5.5] and Proposition 2.3,  $\text{HH}^*(R)$  satisfies the finite generation condition **(fg)** if and only if all  $q_{ij}$  are roots of unity, see [8].

**Proposition 5.9.** *Let  $A$  be a finite dimensional complex (resp. Hopf) algebra that can be defined over some algebraic number field. If the constructed finite dimensional (resp. Hopf) algebra via reduction modulo  $p$  over some finite field is the smash product of a quantum complete intersection with a semisimple Hopf algebra, then  $A$  satisfies **(fg)** (resp. **(hfg)**).*

*Proof.* It follows from Lemma 5.8 and the discussion above, where we notice that any nonzero number in a finite field is a root of unity.  $\square$

Our lifting method of reduction modulo  $p$  can be applied to finite dimensional pointed Hopf algebras of diagonal type over the complex numbers.

*Example 5.10.* Let  $A$  be a finite dimensional pointed Hopf algebra of diagonal type over the field of complex numbers. We follow the strategy of Andruskiewitsch-Schneider in [2, 3]. Denote the coradical of  $A$  by  $\mathbb{C}[G]$  for some finite group  $G$ . Then the associated graded algebra  $\text{gr } A$  is isomorphic to a smash product algebra  $\mathcal{B}(V)\#\mathbb{C}[G]$  with respect to its coradical filtration, where  $\mathcal{B}(V)$  is the Nichols algebra of some Yetter-Drinfeld module  $V$  in  ${}^G\mathcal{YD}$ . Since the braided space  $(V, c)$  is of diagonal type, there is a basis  $\{x_1, \dots, x_\theta\}$  of  $V$  and a collection of scalars  $(q_{ij})_{1 \leq i, j \leq \theta}$  such that  $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$  for all  $1 \leq i, j \leq \theta$ . Notice that the coefficients in the braiding matrix  $\mathbf{q} = (q_{ij})_{1 \leq i, j \leq \theta}$  are all algebraic numbers since they all come from certain characters of the finite group  $G$ . Then one can show that  $\text{gr } A$  can be defined over the algebraic number field  $\mathbb{Q}(q_{ij})$ . Moreover, the Drinfeld double of  $\text{gr } A$ , denoted by  $\mathcal{D}(\text{gr } A)$ , also can be defined over the algebraic number field  $\mathbb{Q}(q_{ij})$ . An explicit presentation of  $\mathcal{D}(\text{gr } A)$  can be found in [43, Lemma 7].

Now suppose the constructed finite-dimensional Hopf algebra from  $\mathcal{D}(\text{gr } A)$  via reduction modulo  $p$  is the smash product of a quantum complete intersection with a semisimple Hopf algebra (e.g.,  $\mathcal{D}(G)$  by a careful choice of  $p$  such that  $p \nmid |G|$ ). Then by Proposition 5.9, we know  $\mathcal{D}(\text{gr } A)$  satisfies **(hfg)**. Moreover by Masuoka's result [29],  $A$  is always some 2-cocycle twist of the associated graded algebra  $\text{gr } A$ . (Also see Angiono and Garcia-Iglesias's survey paper [4, §1.2.1].) As a consequence,  $A$  can be embedded into the Drinfeld double  $\mathcal{D}(\text{gr } A)$ . So we can conclude that  $A$  satisfies **(hfg)** by Proposition 2.8 in this case.

## APPENDIX A. SOME RELEVANT SPECTRAL SEQUENCES RELATED TO COHOMOLOGY

For completeness, we provide here explicit descriptions of the spectral sequences related to the (Hochschild) cohomology of filtered algebras, smash products and crossed products. We follow standard arguments to show that these spectral sequences are multiplicative, that is, they preserve the ring structure of the (Hochschild) cohomology.

**A.1. Cohomology of a filtered augmented algebra.** We recall May's spectral sequence [30] for the cohomology of an augmented filtered algebra. Let  $A$  be an augmented algebra with a finite algebra filtration, either increasing or decreasing:

$$0 = A_{-1} \subset A_0 \subset A_1 \subset \cdots \subset A_m = A \quad \text{or} \quad A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_m = 0,$$

so that  $A_i A_j \subset A_{i+j}$  for all  $i, j$ . Assume the augmentation  $\varepsilon$  on  $A$  satisfies  $\varepsilon(A_n) = 0$  for all  $n > 0$  in the increasing filtration case (in the decreasing filtration case, assume  $\varepsilon(A_1) = 0$ ), so that it induces an augmentation on the associated graded algebra  $\text{gr } A = \bigoplus_{n \geq 0} (\text{gr } A)_n$ . For example, if  $A$  is a Hopf algebra, we will be interested in the cases where either the coradical filtration (increasing) or the Jacobson radical filtration (decreasing) is in fact a Hopf algebra filtration.

The filtration on  $A$  induces a filtration on the reduced bar resolution of  $k$  as an  $A$ -module:

$$(A.1.1) \quad P. : \quad \cdots \xrightarrow{d_3} A \otimes \bar{A}^{\otimes 2} \xrightarrow{d_2} A \otimes \bar{A} \xrightarrow{d_1} A \xrightarrow{\varepsilon} k \rightarrow 0,$$

where  $\bar{A} = A/k \cdot 1_A$  (a vector space quotient). We will give details here for an increasing filtration. A decreasing filtration leads similarly to a spectral sequence. The increasing filtration on  $P_\bullet$  induced by that on  $A$  is given in each degree  $n$  by

$$F_i(A \otimes \bar{A}^{\otimes n}) = \sum_{i_0 + \dots + i_n = i} F_{i_0} A \otimes F_{i_1} \bar{A} \otimes \dots \otimes F_{i_n} \bar{A}.$$

The reduced bar resolution of  $k$  as a module over  $\text{gr } A$  is precisely  $\text{gr } P_\bullet$ , where  $\text{gr}_i P_n := F_i P_n / F_{i-1} P_n$ . Now let  $C^\bullet = C^\bullet(A) := \text{Hom}_A(P_\bullet, k)$ . Then  $C^n(A)$  is a filtered vector space where

$$F^i C^n(A) = \{f \in \text{Hom}_A(P_n, k) : f|_{F_{i-1} P_n} = 0\}.$$

(In the decreasing filtration case,  $F^i C^n(A) = \{f \in \text{Hom}_A(P_n, k) : f|_{F_{i+1} P_n} = 0\}$ .) This filtration is compatible with the differentials on  $C^\bullet(A)$ . Hence,  $C^\bullet(A)$  is a filtered cochain complex. (A decreasing filtration on  $A$  similarly leads to an increasing filtration on  $C^\bullet$ ). By construction, if  $A$  is finite dimensional, the filtration is both bounded above and below. There is a cohomology spectral sequence associated to the filtration on  $C^\bullet$  (see [53, 5.4.1 and 5.5.1]),

$$E_0^{i,j} = \frac{F^i C^{i+j}}{F^{i+1} C^{i+j}}, \quad E_1^{i,j} = H^{i+j}(E_0^{i,*}) \cong H^{i+j}(\text{gr}_i A, k),$$

converging to the cohomology of  $C^\bullet$ :  $H^*(\text{gr}_* A, k) \implies H^*(A, k)$ . The spectral sequence is multiplicative by its definition. Thus  $E_\infty$  is the associated graded algebra of  $H^*(A, k)$ . (See [53, 4.5.2, 5.2.13, and 5.4.8]. See also [31, Theorem 12.5].) If  $M$  is any  $A$ -module, there is an induced filtration with components  $A_i M$  in either case (increasing or decreasing filtration), and there is similarly a spectral sequence  $\tilde{E} := \tilde{E}(M)$  associated to the filtration on  $C^\bullet(A, M) = \text{Hom}_A(P_\bullet, M)$  given by (in the increasing filtration case):

$$F^i C^n(A, M) = \{f \in \text{Hom}_A(P_n, M) : f(F_\ell P_n) \subset F_{\ell-i} M \text{ for all } \ell\}.$$

Then  $\tilde{E}$  is a differential module over  $E$ ,  $\tilde{E}_1^{i,j} \cong H^{i+j}(\text{gr}_i A, \text{gr } M)$ , and  $\tilde{E}$  converges to  $H^*(A, M)$  as an  $H^*(A, k)$ -module (see [30, Theorem 4]).

**A.2. Hochschild cohomology of a filtered algebra.** There are similar spectral sequences for the Hochschild cohomology of a filtered (not necessarily augmented) algebra  $A$ . We assume the filtration on  $A$  is *increasing* (the case of a decreasing filtration is similar). This induces an increasing filtration on the bar resolution  $P_\bullet$  of  $A$  as an  $A$ -bimodule (equivalently,  $A^e$ -module where  $A^e = A \otimes A^{op}$ ). Let  $C^\bullet(A, A) = \text{Hom}_{A^e}(P_\bullet, A)$ , a filtered vector space with *decreasing* filtration:

$$F^i C^n(A, A) = \{f \in \text{Hom}_{A^e}(P_n, A) : f(F_j P_n) \subset F_{j-i} A \text{ for all } j\}.$$

This filtered complex gives rise to a spectral sequence (see [53, 5.4.1 and 5.5.1] or [31]). Similar to the construction given in the augmented algebra setting above,

$$E_0^{i,j} = \frac{F^i C^{i+j}(A, A)}{F^{i+1} C^{i+j}(A, A)}, \quad E_1^{i,j} = H^{i+j}(E_0^{i,*}) \cong \text{HH}^{i+j}(\text{gr}_i A, \text{gr } A).$$

Moreover if  $B$  is an  $A$ -bimodule, the filtration on  $A$  induces a filtration on  $B$  and on  $C^\bullet(A, B) = \text{Hom}_{A^e}(P_\bullet, B)$ :

$$F^i C^n(A, B) = \{f \in \text{Hom}_{A^e}(P_n, B) : f(F_\ell P_n) \subset F_{\ell-i} B \text{ for all } \ell\}.$$

There is similarly a spectral sequence  $\tilde{E}$  associated to this filtration,  $\tilde{E}$  is a differential module over  $E$ ,  $\tilde{E}_1^{i,j} \cong \text{HH}^{i+j}(\text{gr}_i A, \text{gr } B)$ , and  $\tilde{E}$  converges to  $\text{HH}^*(A, B)$  as an  $\text{HH}^*(A)$ -module.

**A.3. Cohomology of an augmented crossed product.** Assume  $H$  is a finite dimensional Hopf algebra with antipode  $S$ ,  $R$  is an algebra for which a crossed product  $R\#_\sigma H$  exists, and  $R$  and  $R\#_\sigma H$  are augmented compatibly with the counit on  $H$ . We explicitly construct a multiplicative spectral sequence converging to  $\text{Ext}_{R\#_\sigma H}^*(k, k)$ . This spectral sequence is known; see e.g. similar constructions of [20, §5] and [49, §2.10]. We will however need some additional structure on the spectral sequences for applications, and we give details of the construction to make these additional structures more transparent.

For any  $R\#_\sigma H$ -modules  $U$  and  $V$ , there is a right  $H$ -module structure on  $\text{Hom}_R(U, V)$  given by

$$(A.3.1) \quad (f \cdot h)(u) = \sum S(h_1) \cdot (f(h_2 \cdot u))$$

for all  $f \in \text{Hom}_R(U, V)$ ,  $h \in H$ , and  $u \in U$ . A calculation shows that  $f \cdot h$  is indeed an  $R$ -module homomorphism for all  $h \in H$  and  $f \in \text{Hom}_R(U, V)$  since  $U, V$  are  $R\#_\sigma H$ -modules. Another calculation shows that this gives an  $H$ -module structure to  $\text{Hom}_R(U, V)$ .

Let  $P_\bullet$  be a projective resolution of  $k$  as an  $H$ -module. Let  $Q_\bullet$  be a projective resolution of  $k$  as an  $R\#_\sigma H$ -module. Since  $R\#_\sigma H$  is free as an  $R$ -module,  $Q_\bullet$  restricts to a projective resolution of  $k$  as an  $R$ -module. Let  $M$  be an  $R\#_\sigma H$ -module, which can be considered as an  $R$ -module.

For all  $i, j \geq 0$ , let

$$C^{i,j}(M) = \text{Hom}_H(P_i, \text{Hom}_R(Q_j, M)).$$

Letting  $H'$  denote homology with respect to the horizontal differentials and  $H''$  with respect to the vertical differentials, the spectral sequence associated to this double complex has second page

$$E_2^{*,*}(M) = H' H''(C^{i,j}) \cong H'(\text{Hom}_H(P_\bullet, H^*(R, M))) \cong H^*(H, H^*(R, M)).$$

We wish to compare with the spectral sequence obtained by reversing the roles of  $i$  and  $j$ . To that end, we will need the following lemma.

**Lemma A.1.** *For each  $j$ , the  $H$ -module  $\text{Hom}_R(Q_j, M)$  is projective.*

*Proof.* For each  $j$ , the  $H$ -module  $Q_j$ , under restriction from  $R\#_\sigma H$ , is projective, since  $R\#_\sigma H$  is free as an  $H$ -module. In order to see that  $\text{Hom}_R(Q_j, M)$  is also projective as an  $H$ -module, it suffices to prove that  $\text{Hom}_R(R\#_\sigma H, M)$  is a projective  $H$ -module, since  $\text{Hom}_R$  and taking cohomology are additive. First notice that

$$\text{Hom}_R(R\#_\sigma H, M) \cong \text{Hom}_k(H, M)$$

as vector spaces since an  $R$ -module homomorphism from  $R\#_\sigma H$  to  $M$  is determined by its values on  $H$ . The right  $H$ -action on  $\text{Hom}_k(H, M)$  induced by this isomorphism is also given by formula (A.3.1). We claim that  $\text{Hom}_k(H, M) \cong M \otimes H^\vee$  as right  $H$ -modules, where  $M$  is a right  $H$ -module via its left  $H$ -module structure and the antipode  $S$ , and  $H^\vee = \text{Hom}_k(H, k)$  is a right  $H$ -module via left multiplication of  $H$  on itself:

$$m \cdot h = S(h) \cdot m \quad \text{and} \quad (f \cdot h)(\ell) = f(h\ell),$$

for all  $m \in M$ ,  $h, \ell \in H$ , and  $f \in H^\vee$ . The isomorphism  $\text{Hom}_k(H, M) \xrightarrow{\sim} M \otimes H^\vee$  is given by sending  $m \otimes f$  to the function taking  $\ell$  to  $f(\ell)m$ , for all  $m \in M$ ,  $f \in \text{Hom}_k(H, M)$ , and  $\ell \in H$ . Since  $H$  is a Frobenius algebra,  $H^\vee \cong H$  as an  $H$ -module (an isomorphism  $H \xrightarrow{\sim} H^\vee$  is given by sending  $h$  to the function that takes  $\ell$  to  $\phi(h\ell)$ , where  $\phi$  is an integral of the dual Hopf algebra to  $H$ ). It follows that  $M \otimes H^\vee \cong M \otimes H$  as  $H$ -modules. Since  $H$  is free as a right  $H$ -module, the  $H$ -module  $M \otimes H$  is projective [6, Proposition 3.1.5].  $\square$

Now, reversing the roles of  $i$  and  $j$ , there is another spectral sequence  $\tilde{E}^{*,*}(M)$  with second page

$$\tilde{E}_2^{*,*}(M) \cong H'' H'(C^{i,j}) = H''(H^*(H, \text{Hom}_R(Q_\bullet, M))).$$

Since  $\text{Hom}_R(Q_j, M)$  is a projective  $H$ -module for each  $j$ , it is also injective, and so

$$\mathrm{H}^*(H, \text{Hom}_R(Q_\bullet, M)) \cong \mathrm{H}^0(H, \text{Hom}_R(Q_\bullet, M)) \cong \text{Hom}_R(Q_\bullet, M)^H \cong \text{Hom}_{R\#_\sigma H}(Q_\bullet, M).$$

Thus we see that  $\tilde{E}_2(M) \cong \mathrm{H}^*(R\#_\sigma H, M)$ .

The above discussion yields the following result.

**Lemma A.2.** *Let  $M$  be an  $R\#_\sigma H$ -module. There is a spectral sequence  $E^{*,*}(M)$  with*

$$E_2^{p,q}(M) = \mathrm{H}^p(H, \mathrm{H}^q(R, M)) \implies \mathrm{H}^{p+q}(R\#_\sigma H, M).$$

*Proof.* By construction,  $E^{*,*}(M)$  is a first quadrant spectral sequence, and so the standard filtration for a double complex is bounded. It follows that  $E^{*,*}(M)$  converges (see, e.g., [53, Theorem 5.5.1]). The rest of the statement results from our discussion above.  $\square$

Next we will show that when  $M = k$ , the spectral sequence  $E^{*,*}(k)$  is multiplicative, and further that  $E^{*,*}(M)$  is a differential bigraded module over  $E^{*,*}(k)$ . We will prove the latter statement; the former is a special case.

For all  $i, j \geq 0$ , let

$$B^{i,j} = \text{Hom}_H(P_i, \text{Hom}_R(Q_\bullet, Q_\bullet)_j),$$

which is quasi-isomorphic to  $C^{\bullet,\bullet}$  when  $M = k$ . There is an action of  $B^{\bullet,\bullet}$  on  $C^{\bullet,\bullet}$  given as follows. Let  $\Delta : P_\bullet \rightarrow P_\bullet \otimes P_\bullet$  be a diagonal chain map. Let

$$\mu : \text{Hom}_R(Q_\bullet, M) \otimes \text{Hom}_R(Q_\bullet, Q_\bullet) \rightarrow \text{Hom}_R(Q_\bullet, M)$$

denote composition of functions.

Let  $f \in C^{i,j}$  and  $g \in B^{i',j'}$ , and define  $f \cdot g \in C^{i+i',j+j'}$  by

$$(A.3.2) \quad f \cdot g = \mu(f \otimes g)\Delta.$$

**Proposition A.3.** *For any  $R\#_\sigma H$ -module  $M$ , the bicomplex  $C^{\bullet,\bullet} = \text{Hom}_H(P_\bullet, \text{Hom}_R(Q_\bullet, M))$  is a differential bigraded module over  $B^{\bullet,\bullet} = \text{Hom}_H(P_\bullet, \text{Hom}_R(Q_\bullet, Q_\bullet))$  under the action defined by equation (A.3.2).*

*Proof.* Let  $f \in C^{i,j}$  and  $g \in B^{i',j'}$ . It suffices for us to check that the following equation holds:

$$(A.3.3) \quad \delta(f \cdot g) = \partial(f) \cdot g + (-1)^{i+j} g \cdot \delta(g),$$

where  $\delta$  is the differential on  $C^{\bullet,\bullet}$  and  $\partial$  is the differential on  $B^{\bullet,\bullet}$ .

To verify (A.3.3), we expand the left side of (A.3.3), letting  $d_Q^*(f \cdot g)$  denote composition of the function  $f \cdot g$  with  $d_Q$ , to obtain

$$\begin{aligned} \delta(f \cdot g) &= d_Q^*(f \cdot g) + (-1)^{i+j+i'+j'+1} (f \cdot g) d_P \\ &= d_Q^* \mu(f \otimes g) \Delta + (-1)^{i+j+i'+j'+1} \mu(f \otimes g) \Delta d_P. \end{aligned}$$

Now  $\Delta$  is a chain map, and  $\text{Hom}_R(Q_\bullet, Q_\bullet)$  is a differential graded algebra with differential graded module  $\text{Hom}_R(Q_\bullet, k)$ , so letting  $\delta_Q$  denote the differential on  $\text{Hom}_R(Q_\bullet, Q_\bullet)$ , the above expression is equal to

$$\begin{aligned} &\mu(d_Q^* f \otimes g) \Delta + (-1)^{i+j} \mu(f \otimes \delta_Q g) \Delta + (-1)^{i+j+i'+j'+1} \mu(f \otimes g) (d_P \otimes 1 + 1 \otimes d_P) \Delta \\ &= \mu(d_Q^* f \otimes g + (-1)^{i+j} f \otimes \delta_Q g + (-1)^{i+j+1} f d_P \otimes g + (-1)^{i+j+i'+j'+1} f \otimes g d_P) \Delta. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \partial(f) \cdot g + (-1)^{i+j} f \cdot \delta(g) \\ &= \mu((d_Q^* f + (-1)^{i+j+1} f d_P) \otimes g) \Delta + (-1)^{i+j} \mu(f \otimes (\delta_Q g + (-1)^{i'+j'+1} g d_P)) \Delta \\ &= \mu(d_Q^* f \otimes g + (-1)^{i+j+1} f d_P \otimes g + (-1)^{i+j} f \otimes \delta_Q g + (-1)^{i+j+i'+j'+1} f \otimes g d_P) \Delta, \end{aligned}$$

and we see that the two expressions are equal, verifying (A.3.3).  $\square$

We now see that from the definitions that each page  $E_r^{*,*}(M)$  is a differential graded module over  $E_r^{*,*}(k)$ . It follows that the action of  $H^*(R\#_\sigma H, k)$  on  $H^*(R\#_\sigma H, M)$  indeed arises as a consequence of the above structure induced on the spectral sequences corresponding to the bicomplexes  $C^{\bullet,\bullet}$  and  $B^{\bullet,\bullet}$ : These structures arise from the diagonal map  $\Delta$  on  $P_\bullet$  and action given by composition on  $\text{Hom}_R(Q_\bullet, M) \otimes \text{Hom}_R(Q_\bullet, Q_\bullet)$ . We conclude a needed result concerning spectral sequences for the cohomology of augmented smash products.

**Proposition A.4.** *Let  $H$  be a finite dimensional Hopf algebra and let  $R$  be an  $H$ -module algebra. Assume  $R$  and  $R\#_\sigma H$  are augmented compatibly with the counit on  $H$ . For any  $R\#_\sigma H$ -module  $M$ , let  $E^{*,*}(M)$  denote the spectral sequence of Lemma A.2. Then  $E^{*,*}(M)$  is a differential bigraded module over the dg algebra  $E^{*,*}(k)$ , and the following properties hold:*

- (i)  $\text{Ext}_R^*(k, k)$  is a graded  $H$ -module algebra.
- (ii)  $\text{Ext}_R^*(k, M)$  is a graded module over  $\text{Ext}_R^*(k, k)\#H$ .

*Proof.* The first statement is an immediate consequence of Proposition A.3 by the above discussion. The rest involves the  $H$ -action. Note that from the definition (A.3.1) of the  $H$ -action, we see that  $\mu$  is an  $H$ -module map, that is  $\mu((f \otimes g) \cdot h) = (\mu \cdot h)(f \otimes g)$ . Further, the action of  $H$  commutes with the differentials. Passing to homology, the induced map

$$H^*(\mu) : H^*(\text{Hom}_R(Q_\bullet, M)) \otimes H^*(\text{Hom}_R(Q_\bullet, Q_\bullet)) \rightarrow H^*(\text{Hom}_R(Q_\bullet, M))$$

is an  $H$ -module map. We see that the following diagram commutes, where the bottom row is given by Yoneda product:

$$\begin{array}{ccc} H^*(\text{Hom}_R(Q_\bullet, M)) \otimes H^*(\text{Hom}_R(Q_\bullet, Q_\bullet)) & \xrightarrow{H^*(\mu)} & H^*(\text{Hom}_R(Q_\bullet, M)) \\ \parallel & & \parallel \\ \text{Ext}_R^*(k, M) \otimes \text{Ext}_R^*(k, k) & \longrightarrow & \text{Ext}_R^*(k, M) \end{array}$$

The diagram is commutative by a standard argument: In the correspondence of elements of  $\text{Ext}$  with generalized extensions, Yoneda composition (or splice) of generalized extensions corresponds with composition of chain maps representing the generalized extensions. Moreover, the  $H$ -action is compatible with the cup/Yoneda products by their definitions.

Since  $\text{Ext}_R^*(k, M) \cong H^*(\text{Hom}_R(Q_\bullet, M))$  and  $\text{Ext}_R^*(k, k) \cong H^*(\text{Hom}_R(Q_\bullet, Q_\bullet))$ , statement (ii) follows. Statement (i) is just a special case by letting  $M = k$ .  $\square$

**A.4. Hochschild cohomology of a smash product.** Let  $H$  be a Hopf algebra with bijective antipode and let  $R$  be an  $H$ -module algebra. Let  $M$  be an  $R\#H$ -bimodule. Ştefan [48] constructed a spectral sequence

$$(A.4.1) \quad H^p(H, \text{HH}^q(R, M)) \implies \text{HH}^{p+q}(R\#H, M),$$

and in fact he did this in the more general setting of a Hopf Galois extension. Taking  $M = R\#H$ , one would like the spectral sequence to be multiplicative. However, Negron [34] pointed out that

it is not multiplicative, and instead gave a different spectral sequence, in this special case of a smash product, that is multiplicative when  $M = R\#H$ . We will show further that the techniques of [34] imply that this alternative spectral sequence is a differential bigraded module over that for  $M = R\#H$ . We first recall the construction of these spectral sequences from [34].

As in [34, Definition 3.1], an  $R$ -bimodule  $U$  is  $H$ -**equivariant** if it is an  $H$ -module for which the structure maps  $R \otimes U \rightarrow U$  and  $U \otimes R \rightarrow U$  are  $H$ -module homomorphisms. As is pointed out in [34], Kaygun [24, Lemma 3.3] showed that  $H$ -equivariant  $R$ -bimodules are in fact modules for a particular smash product  $R^e\#H$ . An  $H$ -equivariant  $R^e$ -complex is defined similarly.

For any  $H$ -equivariant  $R$ -bimodule  $U$  and  $R\#H$ -bimodule  $V$ , there is a right  $H$ -module structure on  $\text{Hom}_{R^e}(U, V)$  given in [34, Definition 4.1]:

$$(f \cdot h)(u) = \sum S(h_1)f(h_2 \cdot u)h_3$$

for all  $f \in \text{Hom}_{R^e}(U, V)$ ,  $h \in H$ , and  $u \in U$ .

Let  $L_\bullet$  be a projective resolution of  $k$  as an  $H$ -module. Let  $K_\bullet$  be an  $H$ -equivariant  $R$ -bimodule resolution of  $R$ . (Existence is discussed in [34]; there it is assumed that  $K_\bullet$  is free over  $R^e$  on a graded base space  $\overline{K}_\bullet$  that is an  $H$ -submodule.) By [34, Corollary 4.4], there is a graded vector space isomorphism

$$\text{HH}^*(R\#H, M) \cong \text{H}^*(\text{Hom}_H(L_\bullet, \text{Hom}_{R^e}(K_\bullet, M))).$$

Standard filtrations of this bicomplex result in a spectral sequence where

$$(A.4.2) \quad \text{H}^p(H, \text{HH}^q(R, M)) \implies \text{HH}^{p+q}(R\#H, M).$$

By [34, Corollary 6.8], when  $M = R\#H$ , the spectral sequence (A.4.2), as constructed above, is multiplicative. The proof of [34, Corollary 6.8] begins with [34, Proposition 6.1], and we extend this to the module setting next. The action of  $\text{Hom}_{R^e}(K_\bullet, R\#H)$  on  $\text{Hom}_{R^e}(K_\bullet, M)$  is defined via a diagonal map  $\omega : K_\bullet \rightarrow K_\bullet \otimes_R K_\bullet$ , analogously to the cup product defined in [34, Section 5], making it a differential graded module.

**Proposition A.5.** *For any  $R\#H$ -bimodule  $M$ , the complex  $\text{Hom}_{R^e}(K_\bullet, M)$  is a differential graded module over  $\text{Hom}_{R^e}(K_\bullet, R\#H)$ , with compatible  $H$ -action.*

*Proof.* This is essentially the proof of [34, Proposition 6.1], checking compatibility with the  $H$ -action.  $\square$

We now conclude a needed result concerning spectral sequences for the Hochschild cohomology of smash products.

**Proposition A.6.** *Let  $H$  be a Hopf algebra with bijective antipode and let  $R$  be an  $H$ -module algebra. For any  $R\#H$ -bimodule  $M$ , let  $E^{*,*}(M)$  denote the spectral sequence (A.4.2) above. Then  $E^{*,*}(M)$  is a differential bigraded module over the dg algebra  $E^{*,*}(R\#H)$  and the following properties hold:*

- (i)  $\text{Ext}_{R^e}^*(R, R\#H)$  is a graded  $H$ -module algebra.
- (ii)  $\text{Ext}_{R^e}^*(R, M)$  is a graded module over  $\text{Ext}_{R^e}^*(R, R\#H)\#H$ .

*Proof.* A similar calculation to that of Proposition A.3 shows directly that  $\text{Hom}_H(L_\bullet, \text{Hom}_{R^e}(K_\bullet, M))$  is a differential bigraded module over  $\text{Hom}_H(L_\bullet, \text{Hom}_{R^e}(K_\bullet, R\#H))$ . Alternatively, we give an argument similar to that in [34] for the first statement. That is, we use Proposition A.5 combined with properties of the resolution  $L_\bullet$  of  $k$  as an  $H$ -module: For any right  $H$ -module  $U$ , let  $U^\uparrow$  denote the induced  $H^e$ -module  $U \otimes_H H^e$  from  $U$ , where  $H^e$  is viewed as a left  $H$ -module with the action given by  $h \mapsto \sum h_1 \otimes S(h_2)$ . Let  $L_\bullet^\uparrow$  denote the complex  $L_\bullet$  for which each module has been induced in this way to an  $H^e$ -module. By [34, Theorem 3.5],  $K_\bullet\#L_\bullet^\uparrow$  is a projective  $R\#H$ -bimodule

resolution of  $R\#H$ . Furthermore, for any  $R\#H$ -bimodule  $M$ , there is a natural isomorphism of chain complexes [34, Theorem 4.3],

$$\Xi : \mathrm{Hom}_{(R\#H)^e}(K.\#L.\uparrow, M) \xrightarrow{\cong} \mathrm{Hom}_H(L., \mathrm{Hom}_{R^e}(K., M)).$$

When  $M = R\#H$ , it follows from [34, Theorem 6.5] that  $\Xi$  is an isomorphism of dg algebras. Moreover, similar to the argument given in the proof of [34, Theorem 6.5], one can check that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(K.\#L.\uparrow, R\#H) \otimes \mathrm{Hom}(K.\#L.\uparrow, M) & \xrightarrow{\Xi \otimes \Xi} & \mathrm{Hom}(L., \mathrm{Hom}(K., R\#H)) \otimes \mathrm{Hom}(L., \mathrm{Hom}(K., M)) \\ \downarrow \rho & & \downarrow \rho' \\ \mathrm{Hom}(K.\#L.\uparrow, M) & \xrightarrow{\Xi} & \mathrm{Hom}(L., \mathrm{Hom}(K., M)), \end{array}$$

where  $\rho$  is the module action resulting from the diagonal map on  $K.\#L.\uparrow$  given in [34, Proposition 6.4] and  $\rho'$  is a similarly defined module action. Thus the action on the left corresponds to the action on the right. The action on the left arises from a diagonal map on  $K.\#L.\uparrow$  and so it is a differential bigraded module structure. Therefore the same is true on the right. The corresponding spectral sequences are induced by the row and column filtrations on the first quadrant double complexes  $\mathrm{Hom}_H(L., \mathrm{Hom}_{R^e}(K., R\#H))$  and  $\mathrm{Hom}_H(L., \mathrm{Hom}_{R^e}(K., M))$ . Thus,  $E^{*,*}(M)$  is a differential bigraded module over the dg algebra  $E^{*,*}(R\#H)$ .

Now taking homology in Proposition A.5, we see that  $\mathrm{Ext}_{R^e}^*(R, M) \cong \mathrm{H}^*(\mathrm{Hom}_{R^e}(K., M))$  is a graded module over  $\mathrm{Ext}_{R^e}^*(R, R\#H) \cong \mathrm{H}^*(\mathrm{Hom}_{R^e}(K., R\#H))$ , and the module structure is compatible with the  $H$ -action. Hence  $\mathrm{Ext}_{R^e}^*(R, M)$  is a graded module over the smash product  $\mathrm{Ext}_{R^e}^*(R, R\#H)\#H$ . This proves (ii). Statement (i) is a special case by letting  $M = R\#H$ .  $\square$

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