Week in Review #6 (2.1-2.4, 3.1-3.4)

In general, the probability of an event is \( P(E) = \frac{n(E)}{n(S)} \).

The Multiplication Principle:
Suppose there are \( m \) ways of performing a task \( T_1 \) and \( n \) ways of performing a task \( T_2 \). Then, there are \( mn \) ways of performing the task \( T_1 \) followed by the task \( T_2 \).

A permutation of a set of distinct objects is an arrangement of these objects in a definite order.

For any natural number \( n \), \( n \)-factorial is given by
\[
 n! = n(n - 1)(n - 2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \\
0! = 1
\]

The number of permutations of \( n \) distinct objects taken \( n \) at a time:
\[
P(n, n) = n!
\]

The number of permutations of \( n \) distinct objects taken \( r \) at a time:
\[
P(n, r) = \frac{n!}{(n - r)!}
\]

Permutations of \( n \) Objects, Not All Distinct:
Given a set of \( n \) objects in which \( n_1 \) objects are alike and of one kind, \( n_2 \) objects are alike and of another kind, \ldots, and, finally, \( n_r \) objects are alike and of yet another kind so that
\[
n_1 + n_2 + \ldots + n_r = n
\]
then the number of permutations of these \( n \) objects taken \( n \) at a time is given by
\[
\frac{n!}{n_1!n_2!\ldots n_r!}
\]

A combination is a subset of \( r \) objects taken from a set of \( n \) objects, without any regard to the order in which the objects are selected.

The number of combinations of \( n \) distinct objects taken \( r \) at a time is given by
\[
C(n, r) = \frac{n!}{r!(n - r)!} \quad \text{(where } r \leq n \text{)}
\]

Bernoulli trials has two outcomes that are independent of each other. The two outcomes are designated as “success” or “failure”; the probability of success is designated \( p \) and the probability of failure \( q \). (Note that \( q = 1 - p \)).

Calculator Functions for binomial experiments (Bernoulli trials):
binompdf (#trials, \( p \), #successes desired)
binomcdf (#trials, \( p \), “at most” #successes)

A random variable is a rule that assigns a number to each outcome of a chance experiment. It is a function with domain given by the set of outcomes of a chance experiment and range contained in the set of real numbers. A random variable is called finite discrete if it assumes only finitely many values; infinite discrete if it takes on infinitely many values, which may be arranged in a sequence; continuous if the values it may assume comprise an interval of real numbers.

The probability distribution of a random variable may be given in the form of a formula or displayed in a table that gives the distinct (numerical) values of the random variable and the probabilities associated with these values. A probability distribution of a random variable may be exhibited graphically by means of a histogram. In a histogram, the area of a rectangle associated with a value of a random variable \( X \) gives precisely the probability associated with the value of
The average, or mean, of the \( n \) numbers \( x_1, x_2, \ldots, x_n \) is \( \bar{x} \) (read “x bar”), where

\[
\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}
\]

Let \( X \) denote a random variable that assumes the values \( x_1, x_2, \ldots, x_n \) with associated probabilities \( p_1, p_2, \ldots, p_n \) respectively. Then the expected value of \( X \), \( E(X) \), is given by

\[
E(X) = x_1 p_1 + x_2 p_2 + \ldots + x_n p_n .
\]

The median of a set of numerical data is the middle number when the numbers are arranged in order of size. If there is an even number of entries, the median is the mean of the two middle numbers.

The mode of a set of observations is the observation that occurs most frequently. If the frequency of two observations is the same and greater than the others, then the set is bimodal. If there is no single most frequent or two the same and greater than the others, there is no mode.

Suppose a random variable has the probability distribution

\[
\begin{array}{cccccc}
\mathcal{P}(X = x) & p_1 & p_2 & p_3 & \ldots & p_n \\
X & x_1 & x_2 & x_3 & \ldots & x_n
\end{array}
\]

and expected value \( E(X) = \mu \)

Then the variance of the random variable \( X \) is

\[
\text{Var}(X) = \sigma^2 = p_1(x_1 - \mu)^2 + p_2(x_2 - \mu)^2 + \ldots + p_n(x_n - \mu)^2
\]

The standard deviation of a random variable \( X \), \( \sigma \), is defined by

\[
\sigma = \sqrt{\text{Var}(X)} = \sqrt{p_1(x_1 - \mu)^2 + p_2(x_2 - \mu)^2 + \ldots + p_n(x_n - \mu)^2} ,
\]

where \( x_1, x_2, \ldots, x_n \) denote the values assumed by the random variable \( X \) and \( p_1 = \mathcal{P}(X = x_1), p_2 = \mathcal{P}(X = x_2), \ldots, p_n = \mathcal{P}(X = x_n) \).

Chebychev’s Inequality

Let \( X \) be a random variable with expected value \( \mu \) and standard deviation \( \sigma \). Then, the probability that a randomly chosen outcome of the experiment lies between \( \mu - k\sigma \) and \( \mu + k\sigma \) is at least \( 1 - \frac{1}{k^2} \); that is, \( P(\mu - k\sigma \leq X \leq \mu + k\sigma) \geq 1 - \frac{1}{k^2} \).

The mean of a binomial distributions is \( \mu = np \); the variance is \( \text{Var}(X) = npq \).

The graph of a normal distribution, which is bell shaped, is called a normal curve. The normal curve (and therefore the corresponding normal distribution) is completely determined by its mean \( \mu \) and standard deviation \( \sigma \). The normal curve has the following characteristics:

1. The curve has a peak at \( x = \mu \).
2. The curve is symmetric with respect to the vertical line \( x = \mu \).
3. The curve always lies above the \( x \)-axis but approaches the \( x \)-axis as \( x \) extends indefinitely in either direction.
4. The area under the curve is 1.
5. For any normal curve, 68.27\% of the area under the curve lies within 1 standard deviation of the mean (that is, between \( \mu - \sigma \) and \( \mu + \sigma \)), 95.45\% of the area lies within 2 standard deviations of the mean, and 99.73\% of the area lies within 3 standard deviations of the mean.
1. How many four-letter code words can be constructed from the first 15 letters of the alphabet if no repetitions are allowed?

2. How many ways can 5 books by one author, 3 by a second, and 6 by a third author be arranged on a shelf, grouping the books together by author?

3. Seven men and eight women are to line up for a picture with all the men in the middle. How many ways can this be done?

4. A box contains 8 red balls, 6 green balls, and 9 black balls. A sample of 10 is to be picked from the box. In how many ways can exactly 5 red balls or exactly 4 green balls be selected?

5. In how many ways can six people boarding a bus be seated if the bus has 8 vacant seats?

6. Twelve students have applied for 4 scholarships from one company. In how many ways can the scholarships be awarded if one particularly student will be given a scholarship?

7. An urn has 8 balls that are identical except that 3 are green and 5 are white. A sample of 6 is selected randomly without replacement. What is the probability that at least 4 of the balls are white?

8. Six people are asked to choose an integer between 1 and 12 inclusive. What is the probability that exactly 4 of the people pick an odd number? Round your answer to 4 decimal places.

9. A 3-card hand is drawn from a standard deck of 52 cards. Find the probability that the hand contains two of the same suit.

10. An exam consists of 25 true/false questions. If a student guesses at every answer, what is the probability that he or she will answer none of them correctly? Round your answer to four decimal places.
11. A shelf at a dollar store contains 70 vases. Nine of the vases are defective. If a customer selects 5 vases at random from the shelf, what is the probability that at least 1 is defective? Round your answer to 4 decimal places.

12. The probability that a certain volleyball player will serve the ball within bounds is 2/5. What is the probability that she will serve exactly 6 balls out of bounds out of 10 serves? Assume the serves are independent. Round your answer to 4 decimal places.

13. The manager of an electronics store will accept a shipment of transistors if there is no more than 1 defective transistor in a random sample of size 30. What is the probability that she will accept the shipment if 10% of the transistors are defective? Round your answer to 4 decimal places.

14. Let $X =$ the number of hours a violinist practices on a given day. Give the range of values the random variable $X$ may assume. What type of random variable is represented?

15. Cards are selected one at a time without replacement from a well-shuffled deck of 52 cards until a spade is drawn. Let $X$ denote the random variable that gives the number of cards drawn. What values may $X$ assume? Classify the random variable type.

16. Let $X$ denote the random variable that gives the number of hearts when 4 cards are drawn from a well-shuffled deck of 52 playing cards. Find $P(X \geq 2)$. Round your answer to 4 decimal places.

17. The probability distribution of the random variable $X$ is shown in the table below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(X = x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>0.15</td>
</tr>
<tr>
<td>-2</td>
<td>0.12</td>
</tr>
<tr>
<td>-1</td>
<td>0.28</td>
</tr>
<tr>
<td>0</td>
<td>0.18</td>
</tr>
<tr>
<td>1</td>
<td>0.11</td>
</tr>
<tr>
<td>2</td>
<td>0.14</td>
</tr>
</tbody>
</table>

Find the following.
(a) $P(X \leq 0)$
(b) $P(X \leq -3)$
(c) $P(-2 \leq X \leq 2)$
18. A box has 5 yellow, 6 blue, and 3 green balls. If 4 balls are drawn at the same time, and $X$ is the number of green balls drawn, compute the following. Round your answer to 4 decimal places.
   a) $P(X = 3)$
   b) $P(X \leq 2)$

19. Let $X$ denote the random variable given by the number of boys in a family of six children. Find the probability distribution (girls and boys are equally likely). Round your answers to 4 decimal places.

<table>
<thead>
<tr>
<th>Random variable, $x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Draw a histogram.

20. Find the expected value $E(X)$ of a random variable $X$ having the following probability distribution. (Enter your answer to two decimal places.)

   $E(X) =$

   $x$  
   $P(X = x)$
   
   -5 0.20  
   -3 0.18  
   -1 0.12  
   1 0.15  
   3 0.16  
   5 0.19  

21. A box has 3 red balls, 5 blue balls, 2 green balls, and 4 yellow balls. A sample of 4 balls is to be selected from the box. Let the random variable $X$ be the number of blue balls drawn in the sample. Give the probability distribution of the random variable $X$. Compute the expected number of blue balls drawn. Give your answer to 4 decimal places.

   $X$  
   $P(X = x)$
   
   0 0.20  
   1 0.40  
   2 0.10  
   3 0.10  
   4 0.20  

22. The probability distribution of a random variable $X$ is given below. Compute the mean, variance, and standard deviation of $X$. Round your answer to 2 decimal places.

   $x$  
   $P(X = x)$
   
   300 0.20  
   330 0.40  
   250 0.10  
   500 0.10  
   440 0.20  

   $E(X) =$

   $Var(X) =$

   $SD(X) =$
23. A probability distribution has a mean of 72 and a standard deviation of 2.5. Use Chebychev’s inequality to estimate the probability that an outcome of the experiment lies between 70 and 80.

24. Find the value of the probability of the standard normal variable $Z$ corresponding to the shaded area under the standard normal curve. Round your answer to 4 decimal places.
   a) $P(Z < 1.35)$
   b) $P(Z > 1.5)$
   c) $P(.85 < Z < 1.6)$

25. $X$ is a continuous random variable that is normally distributed with $\mu = 54$ and $\sigma = 15$. Find the value of $A$ is we know that $P(54 < x < B) = 0.367$ and $P(A < x < B) = 0.843$. Round your final answer to 2 decimal places.

26. Find $P(76 \leq x \leq 103)$ when $\mu = 50$, $\sigma = 40$ for a random variable $x$ with a normal distribution. Round your answer to 4 decimal places.

27. The scores on an exam are normally distributed with a mean of 72 and a standard deviation of 23. The instructor has decided to assign a grade of A to the top 12% of the class and a grade of B to the next 15% of the class. What is the lowest score a student may have on the exam and still obtain a B? Give your answer to 2 decimal places.