DESCRIBING AMOEbas

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Abstract. An amoeba is the image of a subvariety of an algebraic torus under the logarithmic moment map. We consider some qualitative aspects of amoebas, establishing results and posing problems for further study. These problems include determining the dimension of an amoeba, describing an amoeba as a semi-algebraic set, and identifying varieties whose amoebas are a finite intersection of amoebas of hypersurfaces. We show that an amoeba that is not of full dimension is not such a finite intersection if its variety is nondegenerate and we describe amoebas of lines as explicit semi-algebraic sets.

1. Introduction

Hilbert’s basis theorem implies that an algebraic variety is the intersection of finitely many hypersurfaces. A tropical variety is the intersection of finitely many tropical hypersurfaces [4, 8]. These results are important algorithmically, for they allow classical and tropical varieties to be represented and manipulated on a computer. Amoebas and coamoebas are other objects from tropical geometry that are intermediate between classical and tropical varieties, but less is known about how they may be represented.

The amoeba of a subvariety \( V \) of a torus \( (\mathbb{C}^\times)^n \) is its image under the logarithmic moment map \( (\mathbb{C}^\times)^n \to \mathbb{R}^n \) [7, Ch. 6]. The coamoeba is the set of arguments in \( V \). An early study of amoebas [17] discussed their computation. Purbhoo [14] showed that the amoeba of a variety \( V \) is the intersection of amoebas of all hypersurfaces containing \( V \) and that points in the complement of its amoeba are witnessed by certain lopsided polynomials in its ideal. This led to further work on approximating amoebas [18]. Schroeter and de Wolff [15] showed that the amoeba of a point is the intersection of finitely many hypersurface amoebas, and Nisse [10] extended this finiteness to any zero-dimensional variety.

A subvariety of the torus has a finite amoeba basis if its amoeba is the intersection of finitely many hypersurface amoebas. This property is preserved under finite union. We show that a complete intersection of polynomials whose Newton polytopes are affinely independent has a finite amoeba basis and conjecture that finite unions of such varieties are the only varieties having a finite amoeba basis. In support of this conjecture, we show that if a subvariety \( V \subset (\mathbb{C}^\times)^n \) has an amoeba of dimension less than \( n \) and a finite amoeba basis, then each component of \( V \) lies in some translated subtorus.

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Resolving this conjecture may require a better understanding of amoebas. For which we suggest two problems. The first is to show that if the amoeba of $V$ has smaller dimension than expected, $\min\{n, 2\dim_\mathbb{C} V\}$, then $V$ has a particular structure described in Section 4 which explains its dimension. We prove this when the amoeba of $V$ has the minimal possible dimension, showing in that case that $V$ is a single orbit of a subtorus. The second problem asks for a description of the coamoeba and algebraic amoeba (the projection of $V$ to $\mathbb{R}^n_\succ$) of a variety $V$ as semi-algebraic sets. We exhibit such a description for amoebas of lines.

We give some background in Section 2. In Section 3 we observe that coamoebas and algebraic amoebas are semi-algebraic sets and describe the algebraic amoeba of a line as a semi-algebraic set. Section 4 considers the problem of determining the dimension of an amoeba, solving it when the amoeba has minimal dimension. In Section 5 we study when a variety has a finite amoeba basis.

Since this paper appeared on the arXiv, the problem of determining the dimension of an amoeba has been solved by Draisma, Rau, and Yuen [5]. We thank Timo de Wolff, Avgust Tsikh, and Ilya Tyomkin for stimulating conversations, with special thanks to Jan Draisma who pointed out an error in our treatment of dimension in the original version. We also thank the Institute Mittag-Leffler for its hospitality.

2. Amoebas and coamoebas

A point $z$ in the group $\mathbb{C}^\times$ of nonzero complex numbers is determined by its absolute value $|z|$ and its argument $\arg(z)$. These are group homomorphisms that identify $\mathbb{C}^\times$ with the product $\mathbb{R}_\succ \times \mathbb{U}$, where $\mathbb{R}_\succ \subset \mathbb{C}^\times$ is its subgroup of positive real numbers and $\mathbb{U} \subset \mathbb{C}^\times$ is its subgroup of unit complex numbers. The decomposition $\mathbb{C}^\times \simeq \mathbb{R}_\succ \times \mathbb{U}$ induces a decomposition of the algebraic torus $(\mathbb{C}^\times)^n \simeq \mathbb{R}^n_\succ \times \mathbb{U}^n$. Write $| \cdot | : (\mathbb{C}^\times)^n \to \mathbb{R}^n_\succ$ and $\text{Arg} : (\mathbb{C}^\times)^n \to \mathbb{U}^n$ for the projections to each factor. Let $\text{Log} : (\mathbb{C}^\times)^n \to \mathbb{R}^n$ be the composition of the projection $| \cdot |$ with the coordinatewise logarithm.

A subvariety $V \subset (\mathbb{C}^\times)^n$ of the torus the set of zeroes of finitely many Laurent polynomials. It is a hypersurface when it is given by a single polynomial, and a complete intersection if it is given by $r$ polynomials and has dimension $n-r$. The amoeba $\mathcal{A}(V) \subset \mathbb{R}^n$ of a subvariety $V \subset (\mathbb{C}^\times)^n$ of the torus is its image under Log and its coamoeba $\text{co}\mathcal{A}(V) \subset \mathbb{U}^n$ is its image under $\text{Arg}$. Gelfand, Kapranov, and Zelevinsky defined amoebas [7, Ch. 6] and coamoebas first appeared in a 2004 lecture of Passare. The algebraic amoeba $|V| \subset \mathbb{R}^n_\succ$ of a subvariety $V$ is its image under the projection $(\mathbb{C}^\times)^n \to \mathbb{R}^n_\succ$. Because $\mathbb{R}^n_\succ \simeq \mathbb{R}^n$ under the logarithm and exponential maps, $|V| \simeq \mathcal{A}(V)$ as analytic subsets of their respective spaces.

As the map $\mathbb{C}^\times \to \mathbb{R}_\succ$ is proper, the maps $(\mathbb{C}^\times)^n \to \mathbb{R}^n_\succ$ and $(\mathbb{C}^\times)^n \to \mathbb{R}^n$ are proper, and therefore algebraic amoebas and regular amoebas are closed subsets of $\mathbb{R}^n_\succ$ and $\mathbb{R}^n$, respectively. For a single Laurent polynomial $f$, write $\mathcal{A}(f)$ for the amoeba of the hypersurface $\mathcal{V}(f)$ given by $f$. A fundamental geometric fact about amoebas is that each component of the complement $\mathbb{R}^n \setminus \mathcal{A}(f)$ is a convex set [7, Cor. 6.1.6].

The structure of $(\mathbb{C}^\times)^n$ is controlled by its group $\mathbb{Z}^n \simeq N := \text{Hom}(\mathbb{C}^\times, (\mathbb{C}^\times)^n)$ of characters and dual group of characters $\mathbb{Z}^n \simeq M := \text{Hom}((\mathbb{C}^\times)^n, \mathbb{C}^\times)$. For example, Laurent
polynomials are linear combinations of characters. Both \((\mathbb{C}^*)^n\) and \(\mathbb{R}^n \simeq N \otimes_{\mathbb{Z}} \mathbb{R}\) have related structures. A subtorus \(T \subset (\mathbb{C}^*)^n\) corresponds to a saturated subgroup \(\Pi \subset N\) (\(\Pi\) is the set of cocharacters of \(T = \Pi \otimes_{\mathbb{Z}} \mathbb{C}^*\)), as well as to a rational linear subspace \(\Pi_{\mathbb{R}} := \Pi \otimes_{\mathbb{Z}} \mathbb{R}\) of \(\mathbb{R}^n\), which is the amoeba of \(T\). All rational linear subspaces of \(\mathbb{R}^n\) arise in this manner.

A translate \(aT\) of a subtorus \(T\) by an element \(a \in (\mathbb{C}^*)^n\) is an affine subtorus. A rational affine subspace is the amoeba of an affine subtorus, equivalently, it is the translate of a rational linear subspace. A subvariety \(V \subset (\mathbb{C}^*)^n\) is degenerate if it lies in a proper affine subtorus. Otherwise it is nondegenerate. In Section 5, we observe that a variety \(V\) is degenerate if and only if its amoeba lies in a proper rational affine subspace of \(\mathbb{R}^n\).

The logarithmic limit set \(L^\infty(V)\) of a variety \(V \subset (\mathbb{C}^*)^n\) is the set of asymptotic directions of its amoeba, that is, the set of accumulation points of sequences \(\{\frac{z_m}{\|z_m\|}\} \subset S^{n-1}\) where \(\{z_m \mid m \in \mathbb{N}\} \subset \mathcal{A}(V)\) is unbounded. Fundamental work of Bergman [2] and Bieri-Groves [3] show that \(L^\infty(V)\) is the intersection of the sphere \(S^{n-1}\) with a rational polyhedral fan of pure dimension equal to the dimension of \(V\), called the tropical variety of \(V\).

3. Amoebas and coamoebas as semi-algebraic sets

A subset \(X \subset \mathbb{R}^m\) is semi-algebraic if it is defined by finitely many algebraic equations and algebraic inequalities. A semi-algebraic set is basic if it is defined by a finite conjunction of algebraic equations and inequalities. By the Tarski-Seidenberg Theorem, the image of a semi-algebraic set under a polynomial map is again a semi-algebraic set. This has many formulations, a common one is that of quantifier elimination [1, Ch 2]. Since a complex algebraic subvariety \(V \subset (\mathbb{C}^*)^n\) is also a real-algebraic subvariety and the maps \(\cdot\) and \(\text{Arg}\) are real-algebraic maps, the Tarski-Seidenberg Theorem implies the following.

Proposition 3.1. The algebraic amoeba \(|V|\) and the coamoeba co\(\mathcal{A}(V)\) of an algebraic subvariety \(V \subset (\mathbb{C}^*)^n\) are semi-algebraic subsets of \(\mathbb{R}^n\) and \(\mathbb{U}^n\), respectively.

The relative boundary of a semi-algebraic set is a semi-algebraic set. Semi-algebraic sets enjoy finiteness properties; a semi-algebraic set has a finite decomposition into locally closed cells, each of which is a basic semi-algebraic set [1, Ch. 5]. While the amoeba \(\mathcal{A}(V)\) of a variety is not semi-algebraic, it inherits finiteness properties from the algebraic amoeba \(|V|\).

Example 3.2. Let \(a, b, c \in \mathbb{C}^*\). A point \((|x|, |y|)\) lies in the algebraic amoeba of the line \(ax + by + c = 0\) if and only if there is a triangle with sides \(|a||x|, |b||y|, and |c|\). Equivalently,

\[
|a||x| + |b||y| \geq |c| \quad \text{and} \quad ||a||x| - |b||y|| \leq |c|.
\]

This is the shaded polyhedron shown in Figure 1.

The closure \(\overline{\ell}\) of a nondegenerate line \(\ell \subset (\mathbb{C}^*)^3 \subset \mathbb{P}^3\) meets each of the four coordinate planes in \(\mathbb{P}^3\) in a distinct point. As explained in a discussion about coamoebas [12, §3], if the four points of \(\overline{\ell} \setminus \ell\) lie on a circle, then after a reparameterization the line is real, with points in the complement of the circle mapped two-to-one to the amoeba and those on the
circle mapped one-to-one to the relative boundary of the amoeba. If the four points do not lie on a circle, then the map from $\ell$ to the amoeba has no critical points [12, Lem. 7].

Let $\ell \subset (\mathbb{C}^\times)^n$ be a line in that $\overline{\ell} \subset \mathbb{P}^n$ is a line. Set $E_\ell := \ell \setminus \overline{\ell}$ to be the intersection of $\ell$ with the coordinate planes $\mathbb{P}^n \setminus (\mathbb{C}^\times)^n$, its set of ends. Note that $|E_\ell| \geq 2$. The line $\ell$ is real if $E_\ell \subset \ell \simeq \mathbb{P}^1$ lies on a circle and complex if $E_\ell$ does not lie on a circle.

**Lemma 3.3.** A line $\ell \subset (\mathbb{C}^\times)^n$ lies on an affine subtorus $aT_\ell$ of dimension $|E_\ell|-1$ whose closure is a linear subspace of $\mathbb{P}^n$.

**Proof.** Suppose that a point of $E_\ell$ lies in two coordinate planes of $\mathbb{P}^n$, say $x = 0$ and $y = 0$. These coordinates are characters of $(\mathbb{C}^\times)^n$ and they give a coordinate projection to $(\mathbb{C}^\times)^2$. The image of $\ell$ in the $\mathbb{C}^2$ containing this $(\mathbb{C}^\times)^2$ is a line $\ell'$ passing through the origin. Thus $\ell'$ (and therefore $\ell$) satisfies an equation $y = ax$ for some $a \in \mathbb{C}^\times$.

Given a point of $E_\ell$ lying on two or more coordinate planes, let $x_{i_0} = 0, \ldots, x_{i_r} = 0$ be those coordinate planes. These characters $x_{i_j}$ give a coordinate projection to $(\mathbb{C}^\times)^{r+1}$. The image of $\ell$, and thus $\ell$ itself, satisfies an equation $x_{i_j} = a_jx_{i_0}$ for some $a_j \in \mathbb{C}^\times$, for each $j = 1, \ldots, r$. This gives $n+1-|E_\ell|$ independent linear equations that define an affine subtorus $aT_\ell$ of dimension $|E_\ell|-1$ that contains $\ell$. This completes the proof. □

We describe the algebraic amoeba of a line $\ell$ as a semi-algebraic set.

**Theorem 3.4.** Let $\ell \subset (\mathbb{C}^\times)^n$ be a line with algebraic amoeba $|\ell|$ and $aT_\ell$ the affine subtorus containing $\ell$ of Lemma 3.3. When $|E_\ell| = 2$, $\ell = aT_\ell$ and $|\ell|$ is a rational affine line. When $|E_\ell| = 3$, $\ell$ is a nondegenerate line in $aT_\ell \simeq (\mathbb{C}^\times)^2$ and $|\ell|$ is as described in Example 3.2.

Suppose that $|E_\ell| \geq 4$. If $\ell$ is complex, then the map $\ell \to |\ell|$ is a bijection. If $\ell$ is real, then this map is injective on the circle containing $E_\ell$ and two-to-one on its complement.

When $\ell$ is real, $|\ell|$ lies on a surface that is the intersection of $\binom{|E_\ell|}{4}$ quadratic hypersurfaces, one for each projection from $aT_\ell$ to a coordinate $(\mathbb{C}^\times)^3$ and $|\ell|$ is the subset of that surface satisfying inequalities (3.1) from each projection to a coordinate $(\mathbb{C}^\times)^2$.

When $|\ell|$ is complex, $|\ell|$ is the intersection of $A$ quadratic hypersurfaces and $B$ quartic hypersurfaces, where $A$ is the number of subsets of $E_\ell$ of cardinality four that lie on a circle and $B$ is the number of those that do not lie on a circle.
Proof. By Lemma 3.3, if we choose an isomorphism \( aT_\ell \simeq (\mathbb{C}^\times)^{|E_\ell|-1} \) and redefine \( n \), we may assume that \(|E_\ell| = n+1\) and that \( \ell \) meets each coordinate plane in distinct points. The conclusions for \(|E_\ell| < 4\) are immediate. Suppose that \(|E_\ell| \geq 4\). Any equation or inequality satisfied by the image of \(|\ell|\) under a projection to a coordinate subspace is satisfied by \(|\ell|\). Thus the inequalities (3.1) obtained from projections to each coordinate \((\mathbb{C}^\times)^2\) are valid on \(|\ell|\) as are any equations coming from a projection to a coordinate \((\mathbb{C}^\times)^3\). Examples 3.5 and 3.6 show that these equations from each coordinate \((\mathbb{C}^\times)^3\) are quadratic and quartic as the image of the line in that \((\mathbb{C}^\times)^3\) is real or complex, respectively.

We prove the assertions about the degree of the map \( \ell \to |\ell| \). If \( \ell \) is complex, then it is complex in a projection to some coordinate \((\mathbb{C}^\times)^3\). By Example 3.6, the map from \( \ell \) to the algebraic amoeba of such a projection is one-to-one, thus the map \( \ell \to |\ell| \) is one-to-one. When \( \ell \) is real, we may assume that \( E_\ell \subset \mathbb{RP}^1 \subset \ell \) and complex conjugation on \( \mathbb{C} \subset \mathbb{P}^1 = \ell \) is the usual conjugation. Then a point and its conjugate both have the same absolute value, which shows that the map on \( \ell \setminus \mathbb{RP}^1 \) is at least two-to-one. The projection to a coordinate \((\mathbb{C}^\times)^2\) is two-to-one on this set and one-to-one on the real points \( \mathbb{RP}^1 \setminus E_\ell \). This proves the assertion about the degree of \( \ell \to |\ell| \) when \( \ell \) is real.

We show that the necessary inequalities and equations are sufficient to define \(|\ell|\). In Example 3.5, the \( z \)-coordinate of \(|\ell|\) is a function of the \( x \)- and \( y \)-coordinates, as \(|\ell|\) is a graph over its projection to the \((x, y)\)-plane. Thus when \( \ell \) is real, the points of \(|\ell|\) are determined by the quadratic equations from these projections to each coordinate \((\mathbb{C}^\times)^3\), and the inequalities from further projections to each coordinate \((\mathbb{C}^\times)^2\).

When \( \ell \) is complex, at least one projection to a coordinate \((\mathbb{C}^\times)^3\) is a complex line. As shown in Example 3.6, under the further projection to a coordinate \((\mathbb{C}^\times)^2\), this is the graph of two functions, coming from the branches of a quadratic in \( z^2 \). When the projection to a coordinate \((\mathbb{C}^\times)^3\) is real, the previous paragraph shows that the coordinate functions may be recovered from the inequalities and the quadratic equation for this projection. Thus the points of \(|\ell|\) are determined by the quadratic and quartic equations coming from projections to each coordinate \((\mathbb{C}^\times)^3\).

\(\square\)

Example 3.5. We may assume that a real line \( \ell \subset (\mathbb{C}^\times)^3 \) is given by a map

\[ t \mapsto (t, a_2(t - b_2), a_3(t - b_3)) , \]

where \( a_2, a_3, b_2, b_3 \in \mathbb{R}^\times \) with \( b_2 \neq b_3 \). If we set \( t = p + q\sqrt{-1} \) for \((p, q) \in \mathbb{R}^2 \setminus \{(0, 0)\}\), then the algebraic amoeba is the image of the map

\[ \mathbb{R}^2 \ni (p, q) \mapsto \left( \sqrt{p^2 + q^2}, \sqrt{a_2^2(p^2 + q^2 - 2b_2p + b_2^2)}, \sqrt{a_3^2(p^2 + q^2 - 2b_3p + b_3^2)} \right) . \]

Letting \( x, y, z \) be the coordinates for \((\mathbb{R}_\geq)^3\) and squaring gives

\[ x^2 = p^2 + q^2, \quad y^2 = a_2^2(p^2 + q^2 - 2b_2p + b_2^2), \quad \text{and} \quad z^2 = a_3^2(p^2 + q^2 - 2b_3p + b_3^2) . \]

Eliminating \( p \) and \( q \) from these gives the quadratic equation

\[ (b_2 - b_3)x^2 + \frac{b_3}{a_2^2}y^2 - \frac{b_2}{a_3^2}z^2 + b_2b_3(b_2 - b_3) = 0 , \]
that is satisfied by points on the algebraic amoeba. Since projection to any coordinate \((x,y)-, (x,z)-, \text{ or } (y,z)-\) plane in \((\mathbb{C}^\times)^3\) gives a line, the inequalities (3.1) must also be satisfied.

The algebraic amoeba of the real line with parametrization \(t \mapsto (t, t+1, t-1)\) is displayed on the left in Figure 2. It is the set of shaded points on the quadric \(2x^2 - y^2 - z^2 + 2 = 0\), which are those points satisfying
\[
|x + y| \geq 1 \quad \text{and} \quad |x - y| \leq 1.
\]
These are the inequalities (3.1) from its projection to \((x,y)\)-plane, which is the algebraic amoeba of the projection of \(\ell\) to the corresponding coordinate \((\mathbb{C}^\times)^2\).

![Figure 2. Algebraic amoebas of real and complex lines in \((\mathbb{C}^\times)^3\).](image)

**Example 3.6.** Now consider a complex line \(\ell \subset (\mathbb{C}^\times)^3\). After a change of coordinates, we may assume that \(E_\ell = \{0, 1, \infty, \alpha\}\), where \(\alpha\) is not real. Then \(\ell\) has a parametrization
\[
t \mapsto (t, c(t-1), d(t-\alpha)),
\]
for some \(c, d \in \mathbb{C}^\times\). Rescaling the last two coordinates (dividing by \(c\) and \(d\), respectively) the parametrization becomes \(t \mapsto (t, t-1, t-\alpha)\). Writing \(\alpha = -a-b\sqrt{-1}\) with \(a, b \in \mathbb{R}\) and \(b \neq 0\), if \((x, y, z)\) is the point on \(|\ell|\) corresponding to \(t = p+q\sqrt{-1}\), then
\[
x^2 = p^2 + q^2, \quad y^2 = (p-1)^2 + q^2, \quad \text{and} \quad z^2 = (p+a)^2 + (q+b)^2.
\]
Then we have
\[
ay^2 + z^2 - (a+1)x^2 - a^2 - b^2 - a = 2bq \quad \text{and} \quad x^2 - y^2 + 1 = 2p.
\]
It follows that
\[
(3.2) \quad (ay^2 + z^2 - (a+1)x^2 - a^2 - b^2 - a)^2 + b^2(x^2 - y^2 + 1)^2 - 4b^2x^2 = 0.
\]
Thus \(|\ell|\) lies on the part of the quartic surface \(Q \subset \mathbb{R}^n\) defined by (3.2) in the positive orthant. Write \(Q^+\) for this positive part.

We claim that \(Q^+ = |\ell|\). As (3.2) contains only even powers of \(z\), the projection of the quartic \(Q\) to the \((x,y)\)-plane factors through the map \((x,y,z) \mapsto (x,y,z^2)\). The image of \(Q\) is a surface \(S\) on which \(w := z^2\) is a quadratic function of \(x\) and \(y\) with discriminant
\[
\Delta = -4b^2(x - y - 1)(x - y + 1)(x + y - 1)(x + y + 1).
\]
In the quadrant where \( x, y \geq 0 \), this discriminant is positive on the polyhedron \( P \) defined by \( |x + y| \geq 1 \) and \( |x - y| \leq 1 \), which is the algebraic amoeba of the projection of \( ℓ \) to the \((x, y)\)-plane. Thus the surface \( S \) has two branches above points in the interior of \( P \). Note that every point of \( S \) with a positive third coordinate \( w \) gives two points \( \pm \sqrt{w} \) on \( Q \) with exactly one on \( Q^+ \). We claim that on both branches of \( S \), \( w \) is nonnegative and thus that \( Q^+ \) has two branches (and \( Q \) has four branches) above \( P \).

For this, we show that \( |ℓ| \subset Q^+ \) has two points over all interior points of \( P \), except one. Then the composition \( ℓ \to |ℓ| \subset Q^+ \to P \), together with [12, Lem. 7] which asserts that \( ℓ \to |ℓ| \) has no critical points, shows that \( |ℓ| = Q^+ \).

Indeed, let \( t \in \mathbb{C} \setminus \mathbb{R} \). Then \( t = p + q\sqrt{-1} \) with \( q \neq 0 \), and we have

\[
|ℓ(t)| = \left( \sqrt{p^2 + q^2}, \sqrt{(p - 1)^2 + q^2}, \sqrt{(a + p)^2 + (b + q)^2} \right),
\]

These have the same image in the interior of \( P \), but different third coordinates. When \( t = -α \) or \( t = -\overline{α} \), one has third coordinate 0 and does not lie on \( |ℓ| \). Thus \( |ℓ| = Q^+ \).

Let \( ζ \) be a primitive third root of unity. The algebraic amoeba of the symmetric complex line with parametrization \( t \mapsto (t - 1, t - ζ, t - ζ^2) \) is

\[
x^4 + y^4 + z^4 - (x^2 y^2 + x^2 z^2 + y^2 z^2) - 3(x^2 + y^2 + z^2) + 9 = 0 .
\]

Its closure meets the coordinate planes in the (singular) points \((\sqrt{3}, \sqrt{3}, 0)\), \((\sqrt{3}, 0, \sqrt{3})\), and \((0, \sqrt{3}, \sqrt{3})\), and its projection to the \((x, y)\)-plane is determined by the inequalities \( |x + y| \geq 2\sqrt{3} \) and \( |x - y| \leq 2\sqrt{3} \). Two views of this algebraic amoeba and its projection to the \((x, y)\)-plane are shown on the right in Figure 2.

This brings us to our first question. We believe that the semi-algebraic nature of amoebas and coamoebas has been neglected.

**Question 3.7.** Given an algebraic subvariety \( V \subset (\mathbb{C}^*)^n \), produce a description of its algebraic amoeba \( |V| \) and coamoeba \( \cos V \) as semi-algebraic subsets of \( \mathbb{R}^n_+ \) and \( \mathbb{U}^n \).

Besides amoebas of lines, the coamoebas described in [11, §3], and those coming from discriminants [6, 9, 13], we know of no other instances where such a semi-algebraic description has been given. Below, we show the algebraic amoeba of the parabola \( y = (x-1)(x-2) \) and the hyperbola \( y = 1 + 1/(x-2) \).
4. The Dimension of an Amoeba

The dimension of an amoeba may be understood in differential-geometric terms. At a smooth point $x$ of a variety $V \subset (\mathbb{C}^*)^n$ of dimension $k$, the rank of the differential $d_x \log$ of the map to the amoeba is $2k-l$, where $l$ is the dimension of the intersection of $T_x V$ with the tangent space of the fiber $U^n$ at $x$. Since $\sqrt{-1} \cdot T_x U^n = T_x \mathbb{R}^n_-$ and $V$ is complex, $d \log$ and $d \arg$ have the same rank on $T_x V$. Thus the amoeba and coamoeba of $V$ have the same dimension. We would like to understand the dimension of $A(V)$ from the geometry of $V$.

We have the bound $\dim_{\mathbb{R}} A(V) \leq \min\{n, 2 \dim_{\mathbb{C}} V\}$ as $A(V) \subset \mathbb{R}^n$ and $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V$, and we seek structures on $V$ that imply this inequality is strict. If a subvariety $V \subset (\mathbb{C}^*)^n$ has an action by a subtorus $T$ of dimension $l$, then the orbit space $V/T$ is a subvariety of the quotient torus $(\mathbb{C}^*)^n/T$. The amoeba $A(V) \subset \mathbb{R}^n$ has a translation action by the $l$-dimensional rational subspace $A(T)$ with orbit space $A(V/T)$. Taking this into account, we conclude that $\dim_{\mathbb{R}} A(V) \leq \min\{n, 2 \dim_{\mathbb{C}} V - l\}$.

If $V$ lies in an affine subtorus $aT$, then its amoeba lies in $\text{Log}(a) A(T)$, a rational affine subspace of the same dimension as $T$. This further bounds $\dim_{\mathbb{R}} A(V)$.

We identify a structure on $V$ that generalizes these structures. We write $\dim X$ for the dimension of a complex variety $X$ and reserve $\dim_{\mathbb{R}}$ for dimension as a real analytic set.

**Definition 4.1.** Let $V \subset (\mathbb{C}^*)^n$ be an irreducible subvariety and $T \subset (\mathbb{C}^*)^n$ a subtorus. We say that $T$ has a **diminishing action** on $V$ if we have

$$\dim T < 2(\dim V - \dim W) \quad \text{and} \quad 2 \dim W < n - \dim T,$$

where $W := (T \cdot V)/T$ is the image of $V$ in the quotient torus $(\mathbb{C}^*)^n/T$. 

Note that a general fiber of the map $V \to W$ lies in an affine subtorus $aT$. The inequalities imply that the fiber $F$ has small codimension in $aT$ and $W$ has large codimension in $(\mathbb{C}^*)^n/T$.

**Example 4.2.** If $V \subset (\mathbb{C}^*)^n$ lies in a proper affine subtorus $aT$, then $W = (T \cdot V)/T$ is a point and has dimension zero. If $2 \dim V > \dim T$ we have $\dim_{\mathbb{R}} A(V) \leq \dim T < \min\{n, 2 \dim V\}$ and so $T$ has a diminishing action on $V$. If a nontrivial proper torus $T$ acts on $V$ with $n < 2 \dim V - \dim T$, then $T$ has a diminishing action on $V$.

Let $P \subset T \simeq (\mathbb{C}^*)^3$ be a hypersurface with a three-dimensional amoeba and $\ell \subset T' \simeq (\mathbb{C}^*)^3$ be a nondegenerate line. If we set $V := P \times \ell \subset T \times T'$, then $T$ has a diminishing action on $V$ as in this case $W = \ell$ and $\dim V = \dim T = 3$ but $\dim W = 1$ and $n = 6$ so that the inequalities in Definition 4.1 hold so that $T$ has a diminishing action on $V$. Note that $A(V) = A(\ell) \times A(P)$, so that $\dim_{\mathbb{R}} A(V) = 5 < \min\{n, 2 \dim V\}$. 

**Theorem 4.3.** Let $V \subset (\mathbb{C}^*)^n$ be an irreducible subvariety. If a nontrivial proper subtorus has a diminishing action on $V$, then $\dim_{\mathbb{R}} A(V) < \min\{n, 2 \dim V\}$.

**Proof.** Let $T \subset (\mathbb{C}^*)^n$ be a nontrivial a proper subtorus. A general fiber $F$ of $A(V) \to A(W)$ lies in a translation of $A(T)$ and thus has dimension at most $\dim_{\mathbb{C}}(T)$. Thus

$$\dim_{\mathbb{R}} A(V) \leq \dim_{\mathbb{R}} A(W) + \dim_{\mathbb{R}} F_{\mathbb{R}} \leq 2 \dim_{\mathbb{C}} W + \dim_{\mathbb{C}} T.$$
If $T$ has a diminishing action on $V$, then $\dim_{\mathbb{R}} \mathcal{A}(V) < \min\{n, 2 \dim_{\mathbb{C}} V\}$.

We believe the following is true.

**Conjecture 4.4.** For an irreducible subvariety $V \subset (\mathbb{C}^*)^n$, if $\dim_{\mathbb{R}} \mathcal{A}(V) < \min\{n, 2 \dim V\}$, then there is a nontrivial proper subtorus $T$ of $(\mathbb{C}^*)^n$ having a diminishing action on $V$.

We can only prove this when the dimension of the amoeba is the minimum possible.

**Theorem 4.5.** Let $V \subset (\mathbb{C}^*)^n$ be an irreducible subvariety. If $V$ and its amoeba have the same dimension, then $V$ is an affine subtorus of $(\mathbb{C}^*)^n$.

**Proof.** Suppose first that $V$ is a hypersurface, so that it has dimension $n-1$. Then $\mathcal{A}(V) \subset \mathbb{R}^n$ is a hypersurface. Since each component of $\mathbb{R}^n \setminus \mathcal{A}(V)$ is convex, $\mathcal{A}(V)$ must be a hyperplane, as it bounds every such component. Since the logarithmic limit set of $V$—the set of asymptotic directions of $\mathcal{A}(V)$—is a rational polyhedron in $S^{n-1}$ of dimension $n-2$, $\mathcal{A}(V)$ is a rational affine hyperplane, $\Pi_{\mathbb{R}}$, for some subgroup $\Pi \subset \mathbb{Z}^n$ of rank $n-1$.

Let $T := \Pi \otimes_{\mathbb{Z}} \mathbb{C}^*$ be the corresponding subtorus, and let $W := (T \cdot V)/T$ be the image of $V$ in the quotient $\mathbb{C}^* \simeq (\mathbb{C}^*)^n/T$. The amoeba of $W$ is the image of $\mathcal{A}(V)$ in $\mathbb{R} \simeq \mathbb{R}^n/\Pi_{\mathbb{R}}$. As this is a point and $W$ is irreducible, we conclude that $V$ is a single orbit of $T$.

Now suppose that $V$ is not a hypersurface and set $k := \dim_{\mathbb{R}} \mathcal{A}(V) = \dim V$. For every surjective homomorphism $\varphi: (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^{k+1}$ with $\varphi(V)$ a hypersurface, $\mathcal{A}(\varphi(V)) = \Phi(\mathcal{A}(V))$, where $\Phi$ is the corresponding linear surjection $\mathbb{R}^n \rightarrow \mathbb{R}^{k+1}$. By the previous arguments, $\varphi(V)$ is an affine subtorus of $(\mathbb{C}^*)^{k+1}$, and therefore $V$ lies in an affine subtorus of dimension $n-1$. Doing this for sufficiently many independent homomorphisms $\varphi$ and taking the intersections of the affine subtori of dimension $n-1$ proves the theorem. $\square$

5. MOST AMOEBA DO NOT HAVE A FINITE AMOEBA BASIS

As introduced by Schroeter and de Wolff [15], a subvariety $V \subset (\mathbb{C}^*)^n$ has a **finite amoeba basis** if there exist Laurent polynomials $f_1, \ldots, f_r$ such that

$$\mathcal{A}(V) = \mathcal{A}(f_1) \cap \mathcal{A}(f_2) \cap \cdots \cap \mathcal{A}(f_r).$$

Theorem 5.4 shows that an irreducible nondegenerate variety whose amoeba has dimension less than $n$ does not have a finite amoeba basis, and we expect that it is rare for a variety to have a finite amoeba basis. We exhibit some varieties with a finite amoeba basis and conjecture that they are the only such subvarieties of a torus.

By the distributivity of union over intersection, we have the following lemma.

**Lemma 5.1.** The class of varieties having a finite amoeba basis is closed under finite union.

The **Newton polytope** $P(f)$ of a Laurent polynomial $f$ is the convex hull of the set of exponents of its non-zero monomials. A variety $V \subset (\mathbb{C}^*)^n$ is an **independent complete intersection** if it is the complete intersection of polynomials $f_1, \ldots, f_r$, whose Newton polytopes are affinely independent. Any affine subtorus is an independent complete intersection, as subtori of codimension $r$ are defined by $r$ independent binomials.
Theorem 5.2. An independent complete intersection $V$ has a finite amoeba basis. If $f_1, \ldots, f_r$ are polynomials defining $V$ with affinely independent Newton polytopes, then

\[ \mathcal{A}(V) = \mathcal{A}(f_1) \cap \mathcal{A}(f_2) \cap \cdots \cap \mathcal{A}(f_r). \]

Proof. For each $i = 1, \ldots, r$, multiply $f_i$ by a monomial so that its Newton polytope $P(f_i)$ contains the origin. Let $M_i$ be the saturated sublattice of the character lattice such that for $x \mapsto y$:

\[ x \mapsto \text{independent characters} \]

We claim that the product of the hypersurfaces $\mathcal{A}(V)$ is contained in the inverse image of the product of hypersurfaces amoebas as a consequence of (5.1). For the other direction, suppose that $z = (z^{(1)}, \ldots, z^{(r)})$ with $z^{(i)} \in \mathbb{R}^{a_i}$ for $i = 1, \ldots, r$ is a point in the product of the hypersurfaces amoebas. For each $i = 1, \ldots, r$ let $y^{(i)} \in (\mathbb{C}^\times)^{a_i}$ a point in the hypersurface $V(f_i)$. Then $y := (y^{(1)}, \ldots, y^{(r)})$ lies in the product of the hypersurfaces.

As $\varphi$ is surjective, there is a point $x \in (\mathbb{C}^\times)^n$ with $\varphi(x) = y$. By (5.1), $\varphi^{-1}(y) \subset V$. Applying Log shows that $\varphi^{-1}(z) \subset \mathcal{A}(V)$. This implies the other containment, so that

\[ \mathcal{A}(V) = \varphi^{-1}(\mathcal{A}(f_1) \times \cdots \times \mathcal{A}(f_r)). \]

The conclusion of the theorem holds.

By Lemma 5.1, any finite union of independent complete intersections has a finite amoeba basis. We conjecture that these are the only such varieties.

Conjecture 5.3. If a variety has a finite amoeba basis, then it is a finite union of independent complete intersections.

Irreducible nondegenerate independent complete intersections in $(\mathbb{C}^\times)^n$ have amoebas of dimension $n$. We provide the following evidence for Conjecture 5.3.
Theorem 5.4. A nondegenerate irreducible variety \( V \subset (\mathbb{C}^*)^n \) with an amoeba of dimension less than \( n \) does not have a finite amoeba basis.

For example, a nondegenerate irreducible curve in \((\mathbb{C}^*)^n\) for \( n \geq 3 \) does not have a finite amoeba basis. Theorem 5.4 is a consequence of the following lemma.

Lemma 5.5. Let \( V \subset (\mathbb{C}^*)^n \) be a variety with amoeba of dimension \( d < n \). If \( V \) has a finite amoeba basis, then each component of \( V \) lies in an affine subtorus of dimension at most \( d \).

Proof. If \( V \) has a finite amoeba basis, there are Laurent polynomials \( f_1, \ldots, f_r \) such that
\[
\mathcal{A}(V) = \mathcal{A}(f_1) \cap \mathcal{A}(f_2) \cap \cdots \cap \mathcal{A}(f_r).
\]
Since \( \dim_{\mathbb{R}} \mathcal{A}(V) < n \), it has empty interior, and thus every point \( x \) of \( \mathcal{A}(V) \) lies on the boundary \( \partial \mathcal{A}(f_i) \) of some hypersurface amoeba \( \mathcal{A}(f_i) \). Consequently, for every point \( x \in \mathcal{A}(V) \) there is a nonempty subset \( I(x) \subset [r] := \{1, \ldots, r \} \) such that
\[
I(x) = \{ i \in [r] \mid x \in \partial \mathcal{A}(f_i) \}.
\]
For \( I \subset [r] \), set \( U_I := \{ x \in \mathcal{A}(V) \mid I(x) = I \} \). Then
\[
\mathcal{A}(V) = \bigcup_{\emptyset \neq I \subset [r]} U_I.
\]
As the algebraic amoeba of \( V \) as well as those of the hypersurfaces \( \mathcal{V}(f_i) \) are semi-algebraic, standard finiteness arguments imply that \( \mathcal{A}(V) \) has a dense set \( U \) of points \( x \) such that

(i) \( \mathcal{A}(V) \) is smooth at \( x \),
(ii) there is a positive number \( \epsilon \), depending on \( x \), such that \( \mathcal{A}(V) \cap B(x, \epsilon) \subset U_{I(x)} \),
(iii) for each \( i \in I(x) \), \( x \) is a smooth point of \( \partial \mathcal{A}(f_i) \), and
(iv) if \( W \) is an irreducible component of \( V \) with \( x \in \mathcal{A}(W) \), then \( \mathcal{A}(V) \cap B(x, \epsilon) \subset \mathcal{A}(W) \).

Here, \( B(x, \epsilon) \) is the ball in \( \mathbb{R}^n \) centered at \( x \) of radius \( \epsilon \).

Let \( x \in U \) and set \( I := I(x) \). Let \( B := B(x, \epsilon) \cap \mathcal{A}(V) \subset U_I \) be a neighborhood of \( x \), where \( \epsilon > 0 \) is small enough so that if \( j \not\in I \), then \( B(x, \epsilon) \subset \mathcal{A}(f_j) \). As \( B \subset U_I \), we have
\[
B = B(x, \epsilon) \cap \bigcap_{i \in I} \mathcal{A}(f_i) = B(x, \epsilon) \cap \bigcap_{i \in I} \partial \mathcal{A}(f_i).
\]
For each \( i \in I \), we have \( T_x \mathcal{A}(V) = T_x B \subset T_x \partial \mathcal{A}(f_i) \).

Let \( i \in I \). As every component of \( \mathbb{R}^n \setminus \mathcal{A}(f_i) \) is open and convex, \( T_x \partial \mathcal{A}(f_i) \) is a supporting hyperplane to any component whose closure meets \( x \). Thus \( T_x \partial \mathcal{A}(f_i) \) is disjoint from these components. Consequently, there is a neighborhood of \( x \) in \( T_x \partial \mathcal{A}(f_i) \) that lies in \( \mathcal{A}(f_i) \). As \( T_x \mathcal{A}(V) \subset T_x \partial \mathcal{A}(f_i) \), there is a neighborhood of \( x \) in \( T_x \mathcal{A}(V) \) that lies in \( \mathcal{A}(f_i) \). Intersecting these neighborhoods for \( i \in I \) gives a neighborhood of \( x \) in \( T_x \mathcal{A}(V) \) that lies in \( B \). Replacing \( B \) by this neighborhood, we conclude that \( B \) is an open subset of an affine subspace in \( \mathbb{R}^n \).

Let \( W \subset V \) be an irreducible component whose amoeba contains \( B \). By Lemma 5.6 below, \( W \) lies in an affine subtorus of dimension \( \dim_{\mathbb{R}} \mathcal{A}(W) \leq \dim_{\mathbb{R}} \mathcal{A}(V) = d \). Since \( U \) is dense
Lemma 5.6. Let $V \subset (\mathbb{C}^\times)^n$ be an irreducible subvariety such that $\mathcal{A}(V)$ has a point $x$ in whose neighborhood $\mathcal{A}(V)$ is a $d$-dimensional plane. Then $V$ lies in a $d$-dimensional affine subtorus.

Proof. Translating $V$ (and thus $\mathcal{A}(V)$) if necessary, we may assume that $x = 0$. There is a rational $d$-dimensional plane $\Pi_\mathbb{R}$ such that the projection of $T_x\mathcal{A}(V)$ to $\Pi_\mathbb{R}$ is surjective. Taking a subtorus complementary to $\Pi \otimes_\mathbb{Z} \mathbb{C}^\times \simeq (\mathbb{C}^\times)^d$, we have coordinates $(\mathbb{C}^\times)^n = (\mathbb{C}^\times)^d \times (\mathbb{C}^\times)^{n-d}$ and a decomposition of $\mathbb{R}^n = \mathbb{R}^d \oplus \mathbb{R}^{n-d}$ into rational linear subspaces such that $T_x\mathcal{A}(V)$ is the graph of a map $\Lambda: \mathbb{R}^d \to \mathbb{R}^{n-d}$ (here, $\Pi_\mathbb{R} = \mathbb{R}^d$). That is, points of $T_x\mathcal{A}(V)$ are of the form

$$
(y_1, \ldots, y_d, \ell_1(y), \ldots, \ell_{n-d}(y)) | \ y \in \mathbb{R}^d
$$

where $\ell_i$ are the coordinate functions of $\Lambda$, which are linear forms. By our assumption, $\mathcal{A}(V)$ agrees with $T_x\mathcal{A}(V)$ in a neighborhood of $x$, so there is a neighborhood $U$ of the origin in $\mathbb{R}^d$ such that this set (5.2) restricted to $y \in U$ lies in $\mathcal{A}(V)$.

The exponential map on $\mathbb{R}^n$ sends $\mathcal{A}(V)$ to the algebraic amoeba $|V|$. Thus the set

$$
\{(e^{y_1}, \ldots, e^{y_d}, e^{\ell_1(y)}, \ldots, e^{\ell_{n-d}(y)}) | \ y \in U\}
$$

is a neighborhood of the point 1 in $|V|$. In particular, it is a semi-algebraic set.

If we set $z_i := e^{y_i}$, then the exponential of a linear form becomes

$$
e^{\ell_i(y)} = z_1^{\alpha_{i,1}} \cdots z_d^{\alpha_{i,d}} =: z^{\alpha_i},
$$

where $\ell_i(y) = \alpha_{i,1} y_1 + \cdots + \alpha_{i,d} y_d = \alpha_i \cdot y$. In particular, each monomial $z^{\alpha_i}$ is an algebraic function of $z_1, \ldots, z_d$. This implies that the coefficients/exponents $\alpha_{i,j}$ are rational numbers. If we let $\delta$ be their common denominator and set $t_i := z_i^{1/\delta}$ for $i = 1, \ldots, d$—this is well-defined as each $z_i > 0$—then we may assume that each $\alpha_{i,j}$ is an integer.

Let $\mathbb{T}$ be the $d$-dimensional subtorus of $(\mathbb{C}^\times)^n$ whose algebraic amoeba is the Zariski closure of the set (5.3). That is, $\mathbb{T}$ is defined in $(\mathbb{C}^\times)^n$ by $x_{n+i} = x_1^{\alpha_{i,1}} \cdots x_d^{\alpha_{i,d}}$ for each $i = 1, \ldots, n-d$. Then the image of $V$ in $(\mathbb{C}^\times)^n/\mathbb{T}$ is contained in the compact subtorus of $(\mathbb{C}^\times)^n/\mathbb{T}$, which implies that the image of $V$ is a single point as $V$ is irreducible. This completes the proof.

We close with a simple characterization of degenerate varieties.

Lemma 5.7. An irreducible subvariety $V$ of $(\mathbb{C}^\times)^n$ lies in an affine subtorus of dimension $d$ if and only if its amoeba lies in an affine subspace of $\mathbb{R}^n$ of dimension $d$. 

Proof. Suppose that $\mathcal{A}(V)$ lies in a proper affine subspace $a \mathcal{A}(T)$, for a subtorus $T$ of $(\mathbb{C}^*)^n$. Then the amoeba of the irreducible variety $(T \cdot V)/T$ is a point, which implies that $T \cdot V$ is a single orbit of $T$. Noting that $V \subset aT$ implies that $\mathcal{A}(V) \subset \log(a) \mathcal{A}(T)$, which is an affine subspace of $\mathbb{R}^n$ of dimension $\dim T$, completes the proof. □

References


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