CLASSIFICATION OF SCHUBERT GALOIS GROUPS IN $Gr(4,9)$

ABRAHAM MARTÍN DEL CAMPO, FRANK SOTTILE, AND ROBERT LEE WILLIAMS

Abstract. We classify Schubert problems in the Grassmannian of 4-planes in 9-dimensional space by their Galois groups. Of the 31,806 essential Schubert problems in this Grassmannian, only 149 have Galois group that does not contain the alternating group. We identify the Galois groups of these 149—each is an imprimitive permutation group. These 149 fall into two families according to their geometry. This study suggests a possible classification of Schubert problems whose Galois group is not the full symmetric group, and it begins to establish the inverse Galois problem for Schubert calculus.

INTRODUCTION

In “Traité des substitutions et des équations algébriques”, Jordan [16] explained how a problem in enumerative geometry has a Galois group that acts as a permutation group on its solutions. If the solutions of an enumerative problem possess additional structure, its Galois group must preserve that structure, and thus cannot be the full symmetric group—we call such a problem/Galois group enriched. Jordan studied several enriched enumerative problems, such as the 27 lines on a smooth cubic surface in $\mathbb{P}^3$. Later, Harris [11] showed that each of Jordan’s enriched problems had Galois group as large as possible given its structure. He also showed that natural extensions of each of Jordan’s enriched problems, as well as some other enumerative problems, such as Chasles’ problem of 3264 plane conics tangent to five given conics [5], have full symmetric Galois groups.

Until recently, Galois groups of enumerative problems were difficult to study, for there were few methods available. Vakil’s geometric Littlewood-Richardson rule [25, 26] leads to a recursive method that may show the Galois group of a Schubert problem in a Grassmannian (a Schubert Galois group) contains the alternating group on its solutions (is at least alternating). Numerical algebraic geometry can compute a monodromy group [12, 20], which equals the Galois group [11, 15]. Symbolic computation can compute cycle types of Frobenius elements in Galois groups over $\mathbb{Q}$. With many thousands to hundreds of millions of computable Schubert problems in Grassmannians, Schubert calculus forms a laboratory for studying Galois groups in enumerative geometry.

Write $Gr(k, n)$ for the Grassmannian of $k$-planes in $\mathbb{C}^n$. Vakil [26] used his method to show that every Schubert Galois group in $Gr(2, n)$ for $n \leq 16$ and in $Gr(3, n)$ for $n \leq 9$ is at least alternating. The Galois group of any Schubert problem with two or three solutions is full symmetric. Derksen found an enriched problem in $Gr(4,8)$ with

\begin{thebibliography}{9}

2010 Mathematics Subject Classification. 14N15, 12F10, 12F12.

Key words and phrases. Schubert calculus, Grassmannian, Galois group, permutation group.
Work of Martín del Campo supported in part by CONACyT under grant Cátedra-1076.
Work of Sottile supported in part by the National Science Foundation under grant DMS-1501370.

1

\end{thebibliography}
six solutions, which Vakil generalized to an infinite family of enriched Schubert problems with members in every Grassmannian \( Gr(k, n) \) for \( 4 \leq k \leq n-4 \) \cite[§3.14]{26}. Numerical computation of monodromy showed \cite{20} that a number of simple (explained in §1.1) Schubert problems have Galois group the full symmetric group, including one with 17,589 solutions in \( Gr(3, 9) \). Combinatorial and analytic arguments starting with Vakil’s method showed \cite{3} that, for any \( n \), every Schubert Galois group in \( Gr(2, n) \) is at least alternating. Geometric-combinatorial methods were used in \cite{24} to show that every Schubert Galois group in \( Gr(3, n) \) is 2-transitive. A Schubert problem is essential if it cannot be reduced to one on a smaller Grassmannian. Galois groups of all 3501 essential Schubert problems in \( Gr(4, 8) \) were studied in \cite[§4]{21} and \cite{24}. All except 14 are at least alternating. Each of those 14 have an imprimitive Galois group and they fall into three families with Derksen’s example forming one family.

Here, we study the Galois groups of all 31,806 essential Schubert problems in the next Grassmannian, \( Gr(4, 9) \), determining that all but 149 are at least alternating. These 149 have an imprimitive Galois group that is a wreath product of two symmetric groups. All 149 and 13 of the 14 in \( Gr(4, 8) \) have a common structure—they are a fibration of Schubert problems in either \( Gr(2, 4) \) or \( Gr(2, 5) \), which explains their Galois group. The only enriched problem that is not a fibration is Derksen’s, whose structure was explained by Vakil. The simplicity of this classification for \( Gr(4, 8) \) and \( Gr(4, 9) \) suggests the possibility of classifying all enriched Schubert problems in Grassmannians.

Each Schubert Galois group over \( \mathbb{C} \) is a normal subgroup of the corresponding group over \( \mathbb{Q} \), and we conjecture that the two groups are equal. Computing cycle types of Frobenius elements enables us to determine the Galois group over \( \mathbb{Q} \) of every essential Schubert problem in \( Gr(4, 9) \) with fewer than 300 solutions—26,051 problems in all. Each enriched Schubert problem in \( Gr(4, 9) \) has ten or fewer solutions. Other than the 149 enriched problems, all remaining 26,353 are full symmetric. We highlight this observed dichotomy from our study of all small Grassmannians.

**Theorem 1.** Every known Schubert Galois group is either

1. the full symmetric group on the solutions, and hence maximally transitive, or
2. an imprimitive permutation group, and hence is not 2-transitive.

Recently, Esterov \cite{8} has shown a similar dichotomy for systems of sparse polynomial equations; either their Galois group is full symmetric or it is imprimitive.

The known Schubert Galois groups are very particular permutation groups. Most are symmetric groups \( S_d \) acting naturally on the set \( [d] := \{1, \ldots, d\} \). The Galois group of Derksen’s example is the induced action of \( S_4 \) on the six equipartitions of \( [4] \),

\[
12|34, 13|24, 14|23, 23|14, 24|13, 34|12.
\]

This action is imprimitive as it preserve the partition

\[
\{12|34, 34|12\} \sqcup \{13|24, 24|13\} \sqcup \{14|23, 23|14\}.
\]

Members of Vakil’s infinite family have a similar imprimitive action of \( S_n \) on certain partitions of \([n]\). Lastly, the remaining Galois groups we found are wreath products of
two symmetric groups $S_m \wr S_d = (S_m)^d \rtimes S_d$. In our study we found that each essential enriched Schubert problem in $Gr(4,9)$ has Galois group $S_2 \wr S_2$, $S_3 \wr S_2$, $S_5 \wr S_2$, or $S_2 \wr S_3$.

Mathematicians from Hilbert to Arnold have stimulated the development of mathematics by proposing to address the first nontrivial or next unknown instance of a general question. This investigation is in their spirit. As it touches on geometry, combinatorics, number theory, and group theory, it represents the unity of mathematics.

We begin in Section 1 by sketching some background, including Schubert problems in Grassmannians, Galois groups of branched covers, and permutation groups. In Section 2 we discuss our computations, which give a lower bound for each Schubert Galois group. Section 3 contains a detailed study of the structure of some Schubert problems, identifying classes of enriched Schubert problems whose Galois group is a subgroup of a nontrivial wreath product, and determining the Galois group of each enriched problem in $Gr(4,9)$. This explains general methods to study Schubert Galois groups and it points to a possible classification of enriched Schubert problems in Grassmannians.

Some of this paper is based on the 2017 Ph.D. thesis of Williams [27]. The computations used a Maple script of Vakil\textsuperscript{1}, as well as software developed by the authors and by Christopher Brooks, Aaron Moore, James Ruffo, and Luis García-Puente.

1. Background

We work over the complex numbers, $\mathbb{C}$, although our results hold for any algebraically closed field of characteristic zero.

1.1. Schubert Calculus in Grassmannians. The Schubert calculus in Grassmannians concerns all problems of enumerating the linear subspaces of a vector space that satisfy incidence conditions imposed by other, fixed linear subspaces. The simplest non-trivial Schubert problem asks for the 2-planes in $\mathbb{C}^4$ that meet four general 2-planes nontrivially. Passing to lines in projective 3-space, this asks for the lines that meet four general lines.

To understand the solutions, consider first three pairwise skew lines $\ell_1$, $\ell_2$, and $\ell_3$. They lie on a unique hyperboloid (see Figure 1). This hyperboloid has two rulings by lines: $\ell_1$, $\ell_2$, and $\ell_3$ lie in one, and the lines that meet all three form the second. The fourth line, $\ell_4$, meets the hyperboloid in two points, and the lines $m_1$ and $m_2$ in the second ruling through these points are the two solutions to our problem of four lines.

Since $m_1$ and $m_2$ lie in the same ruling, they do not meet. Figure 1 also illustrates that the Galois group of this Schubert problem is the symmetric group $S_2$. Rotating the line $\ell_4$ $180^\circ$ about the point $p$ interchanges two solution lines, so the Galois group contains a transposition. For another way to see this, observe that if $\ell_4$ varies in a pencil, it will become tangent to the hyperboloid and the two lines will coincide. This implies that the local monodromy in the pencil near the tangent line contains a 2-cycle.

Let $V$ be a complex vector space of dimension $n$. Write $Gr(k,n)$ or $Gr(k,V)$ for the Grassmannian of $k$-planes in $V$. Note that it is an algebraic manifold of dimension $k(n-k)$ [9], and that $Gr(k,n)$ is isomorphic to $Gr(n-k,n)$. The ambient space for the

\textsuperscript{1}http://math.stanford.edu/~vakil/programs/galois
Problem of four lines is $Gr(2, 4)$. Incidence conditions on the $k$-planes of $Gr(k, n)$ are indexed by partitions, which are weakly decreasing sequences of nonnegative integers

$$\lambda: n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0.$$ 

For example, both $(4, 4, 3, 1)$ and $(3, 1, 0, 0)$ are partitions for $Gr(4, 9)$. Trailing 0s are often omitted, and we represent partitions by their Young diagrams, which are left-justified arrays of boxes, with $\lambda_i$ boxes in row $i$. For these partitions, we have

$$(4, 4, 3, 1) = \begin{array}{c|c|c|c} 
| & | & | & |
\end{array} \quad \text{and} \quad (3, 1) = \begin{array}{c|c} 
| & |
\end{array}.$$

More generally, given positive integers $a < b$ and a partition $\lambda$, we say that $\lambda$ is a partition for $Gr(a, b)$ when $\lambda_1 \leq b-a$ and $\lambda_{a+1} = 0$.

A condition $\lambda$ is imposed by a flag, which is a sequence of linear subspaces,

$F_\bullet: \{0\} \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = \mathbb{C}^n,$

where $\dim(F_i) = i$. The condition $\lambda$ on a $k$-plane $H \in Gr(k, n)$ imposed by the flag $F_\bullet$ is

$$\dim H \cap F_{n-k+i-\lambda_i} \geq i \quad \text{for} \quad i = 1, \ldots, k.$$ 

For $\lambda = (3, 1)$ on $Gr(4, 9)$, this is

$$\dim H \cap F_{9-4+1-3} = \dim H \cap F_3 \geq 1 \quad \text{and} \quad \dim H \cap F_{9-4+2-1} = \dim H \cap F_6 \geq 2,$$

as the conditions for the trailing 0s, $\dim H \cap F_8 = 3$ and $\dim H \cap F_9 = 4$, always hold.

The set of all $H \in Gr(k, n)$ satisfying (2) is the Schubert variety $\Omega_\lambda F_\bullet$. This is an irreducible subvariety of $Gr(k, n)$ of codimension $|\lambda| := \lambda_1 + \cdots + \lambda_k$. A Schubert problem in $Gr(k, n)$ is a list $\lambda = (\lambda^1, \ldots, \lambda^s)$ of partitions with $|\lambda^1| + \cdots + |\lambda^s| = k(n-k)$. An instance of a Schubert problem $\lambda$ comes with a specific choice of the flag $F_\bullet$. In the problem of four lines, the condition that a 2-plane $H$ in $\mathbb{C}^4$ meets the 2-dimensional subspace of $F_\bullet$ is encoded by the partition $(1, 0) = \square$. The Schubert variety $\Omega_\lambda F_\bullet$ is a hypersurface in
the four-dimensional Grassmannian $Gr(2, 4)$. The problem of four lines is the Schubert problem $(\square, \square, \square, \square)$, which we write in multiplicative form as $\square^4$; the exponent 4 indicates that $\square$ is repeated four times. A Schubert problem is simple if all except possibly two of its conditions are hypersurface Schubert conditions, $\square$, that is, if it has the form $\lambda \cdot \mu \cdot \square^{s-2}$.

For a Schubert problem $\lambda$, if $\mathcal{F}_\bullet := (F_1^1, \ldots, F_s^s)$ are general flags, the intersection

$$
\Omega_{\lambda} \mathcal{F}_\bullet := \Omega_{\lambda, F_1^1} \cap \Omega_{\lambda, F_2^2} \cap \cdots \cap \Omega_{\lambda, F_s^s}
$$

is transverse [18] and consists of finitely many points. This number $d(\lambda)$ of points is independent of the choice of general flags. It may be computed using algorithms from the Schubert calculus [9, 17]. For brevity, we sometimes denote Schubert problems by $\lambda = d(\lambda)$ to specify the number of solutions to it, e.g., the problem of four lines can be denoted as $\square^4 = 2$, to state that the problem has two solutions. To enumerate all 81,533 nontrivial ($d(\lambda) > 1$) Schubert problems in $Gr(4, 9)$ and compute $d(\lambda)$ takes about 31,806 of the 81,533 Schubert problems are essential.

For some Schubert problems $\lambda$ in $Gr(k, n)$, one or two conditions imply that there is a hyperplane $\Lambda$ (respectively a line $\ell$) such that any solution $H$ must lie in $\Lambda$ (respectively contain $\ell$). In these cases, the Schubert problem $\lambda$ is equivalent to one in the smaller Grassmannian $Gr(k, n-1)$ (respectively $Gr(k-1, n-1)$). By [24, Prop. 6] these two Schubert problems have isomorphic Galois groups. A Schubert problem in $Gr(k, n)$ is essential if it is not equivalent to one in a smaller Grassmannian in this way. On $Gr(4, 9)$ only 31,806 of the 81,533 nontrivial Schubert problems are essential.

The set $\mathbb{F}^s\ell_n$ of all flags in $\mathbb{C}^n$ is a smooth irreducible rational algebraic variety of dimension $\binom{n}{2}$. Given a Schubert problem $\lambda = (\lambda^1, \ldots, \lambda^s)$, we have the incidence variety

$$
\mathcal{X}_\lambda := \{(H, F_1^i, \ldots, F_s^i) \mid H \in \Omega_{\lambda, F_i^i} \text{ for } i = 1, \ldots, s\}.
$$

The map $\mathcal{X}_\lambda \to Gr(k, n)$ is a fiber bundle with reducible fibers, so $\mathcal{X}_\lambda$ is irreducible. For an $s$-tuple $\mathcal{F}_\bullet = (F_1^i, \ldots, F_s^i)$ of flags, the fiber $p^{-1}(\mathcal{F})$ is the intersection $\Omega_{\lambda, \mathcal{F}_\bullet}$ (3).

The variety $\mathcal{X}_\lambda \subset Gr(k, n) \times (\mathbb{F}^s\ell_n)$ may be defined in local coordinates by polynomials with integer coefficients. The dimension conditions (2) are formulated as rank conditions on matrices, which become the vanishing of minors of matrices whose entries are variables or constants. This and other formulations are explained in [13, 14, 19, 21].

1.2. Galois groups of branched covers. Suppose that $\pi : X \to Y$ is a dominant map of complex irreducible varieties of the same dimension. Then there is a positive integer

\footnote{http://www.math.tamu.edu/~sottile/research/stories/GIVIX}
$d$ and a dense open subset $U \subset Y$ consisting of regular values $u$ of $\pi$ whose fiber $\pi^{-1}(u)$ consists of $d$ reduced points. Thus, over $U$, the map $\pi : X|_U \to U$ is a covering space of degree $d$. Call $\pi : X \to Y$ a branched cover of degree $d$.

Since $\pi(X)$ is dense in $Y$ and both varieties are irreducible, there is an inclusion of function fields $\pi^* : \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$. As the map $\pi$ has degree $d$, the field extension $\mathbb{C}(X)/\pi^*\mathbb{C}(Y)$ has degree $d$. Let $\text{Gal}_\pi$ be the Galois group of the Galois closure of $\mathbb{C}(X)$ over $\pi^*\mathbb{C}(Y)$. This is also the monodromy group of the covering space $\pi : X|_U \to U$, which acts transitively on a fiber as $X$ is irreducible. Indeed, for any two points $x, x'$ in a given fiber $\pi^{-1}(u)$ over a point $u \in U$, there is a path in $X|_U$ connecting $x$ to $x'$. Its image in $U$ is a loop based at $u$ whose corresponding monodromy permutation sends $x$ to $x'$. The realization that the algebraic Galois group is the geometric monodromy group goes back at least to Hermite [15], with a modern treatment given in [11].

Branched covers $\pi : X \to Y$ are common in enumerative geometry and are the source of Jordan’s observation that enumerative problems have Galois groups [16]. If $Z \subset Y$ is an irreducible subvariety that meets the regular locus $U$ of $\pi$, then $X|_Z \to Z$ (or rather the closure of $X|_{Z\cap U}$ in $X$) is a branched cover and its monodromy group is a subgroup of $\text{Gal}_\pi$. This holds even if $X|_Z$ is reducible.

When $\pi : X \to Y$ is defined over $\mathbb{Q}$, we may similarly define a Galois group $\text{Gal}_\pi(\mathbb{Q})$ using the Galois closure $L$ of $\mathbb{Q}(X)$ over $\pi^*(\mathbb{Q}(Y))$. This is not necessarily equal to the Galois group $\text{Gal}_\pi$ over $\mathbb{C}$—the difference occurs when $L$ contains scalars, in that $\mathbb{Q} \subsetneq L \cap \mathbb{C}$. For example, when $Y = \mathbb{A}^1$ and $X = Y(x^3 - y)$, then $\text{Gal}_\pi = \mathbb{Z}/3\mathbb{Z}$, but $\text{Gal}_\pi(\mathbb{Q}) = S_3$, the difference being that the primitive third roots of unity lie in $\mathbb{C}$ and not in $\mathbb{Q}$. In general $\text{Gal}_\pi$ is a normal subgroup of $\text{Gal}_\pi(\mathbb{Q})$ and the quotient is the Galois group of the scalar extension in $L/\mathbb{Q}(Y)$. This distinction will be important in Section 2.2.

The Schubert Galois group $\text{Gal}_\lambda$ of a Schubert problem $\lambda$ is the Galois group of the branched cover $\pi : X_\lambda \to (\mathbb{F}\ell_n)^*$ defined in (4). Choosing a regular value $F_\bullet \in (\mathbb{F}\ell_n)^*$ so that $\pi^{-1}(F_\bullet)$ consists of $d(\lambda)$ reduced points, the group $\text{Gal}_\lambda$ is a transitive subgroup of the symmetric group $S_d(\lambda)$. The Grassmannian and flag manifold are rational varieties defined over $\mathbb{Q}$, so we have the group $\text{Gal}_\lambda(\mathbb{Q})$ which contains $\text{Gal}_\lambda$ as a normal subgroup. Based on the results of Sections 2.2 and 3, we make the following conjecture.

**Conjecture 2.** For any Schubert problem $\lambda$, $\text{Gal}_\lambda(\mathbb{Q}) = \text{Gal}_\lambda$.

The following simple proposition, in particular constructions appearing in its proof, will be important to establish the results of Section 3.

**Proposition 3.** The Galois groups of $\square^4 = 2$ in $\text{Gr}(2, 4)$, as well as $\square \cdot \square^3 = 2$, $\square \cdot \square^4 = 2$, $\square^2 \cdot \square^2 = 2$, $\square \cdot \square^4 = 3$, and $\square^6 = 5$ in $\text{Gr}(2, 5)$ are all the corresponding symmetric group.

**Proof.** All Schubert Galois groups in $\text{Gr}(2, n)$ contain the alternating group [3]. We show that each contains a transposition, which will complete the proof.

As these are monodromy groups of branched covers $X_\lambda \to (\mathbb{F}\ell_n)^*$, it will suffice to show there is a choice of flags $F_\bullet$ whose corresponding instance $\Omega_\lambda F_\bullet$ has a unique double solution. We describe a configuration of flags with one flag lying in a pencil such that the pencil of instances contains an instance with a double solution.
For $\square^4 = 2$ in $Gr(2, 4)$, note that only the two-dimensional subspaces of the flags matter. As explained following Figure 1, if we start with a configuration of four flags (lines $\ell_1, \ldots, \ell_4$ in Figure 1) and then let $\ell_4$ move in a pencil, the two members of that pencil that are tangent to the quadric each give an instance with a double solution.

The other Schubert problems in $Gr(2, 5)$ with two solutions are not essential; they reduce to $\square^4 = 2$ in a $Gr(2, 4)$. This completes the proof in these cases, by [24, Prop. 6].

For $\square \cdot \square^3 = 3$, the relevant subspaces are $F_2^5, F_3^5, \ldots, F_5^5$, and the Schubert problem asks for the 2-planes $H$ that have a nontrivial intersection with each. If the first two subspaces are in a degenerate configuration where $\ell := F_2^1 \cap F_2^3$ has dimension 1 so that $\Lambda := \langle F_2^1, F_3^5 \rangle$ has dimension 4, then

$$\Omega_{\square} F_2^1 \cap \Omega_{\square} F_3^5 = \Omega_{\square} \ell \cup \Omega_{\square} (F_2^1 \subset \Lambda),$$

so that the Schubert problem breaks into two subproblems. The one involving $\Omega_{\square} \ell$ has the unique solution $\langle \ell, F_2^3 \rangle \cap \langle \ell, F_3^5 \rangle$, while the one involving $\Omega_{\square} (F_2^1 \subset \Lambda)$ is an instance of $\square \cdot \square^3 = 2$. Letting one of $F_3^3, F_4^3$, or $F_5^3$ move in a pencil proves this case.

Finally, the Schubert problem $\square^6 = 5$ asks for the 2-planes $H$ that meet each of six 3-planes $F_3^1, \ldots, F_5^5$. Suppose that $F_3^1$ and $F_3^2$ are in degenerate position so that $L^{12} := F_3^1 \cap F_3^2$ is a 2-plane and $\Lambda^{12} := \langle F_3^1, F_3^2 \rangle$ is a 4-plane. Supposing also that $F_3^3$ and $F_3^4$ are in a similar degenerate position, then the Schubert problem becomes

$$(\Omega_{\square} \Lambda^{12} \cup \Omega_{\square} L^{12}) \cap (\Omega_{\square} \Lambda^{34} \cup \Omega_{\square} L^{34}) \cap \Omega_{\square} F_5^3 \cap \Omega_{\square} F_6^5.$$

This gives four subproblems, which are the intersection of $\Omega_{\square} F_3^3 \cap \Omega_{\square} F_3^6$ with one of

$$\Omega_{\square} \Lambda^{12} \cap \Omega_{\square} \Lambda^{34}, \quad \Omega_{\square} \Lambda^{12} \cap \Omega_{\square} \Lambda^{34}, \quad \Omega_{\square} \Lambda^{12} \cap \Omega_{\square} \Lambda^{34}, \quad \Omega_{\square} \Lambda^{12} \cap \Omega_{\square} \Lambda^{34}.$$

The first three each have a unique solution—for example the first is the span of the two one-dimensional linear subspaces

$$F_3^3 \cap \Lambda^{12} \cap \Lambda^{34} \quad \text{and} \quad F_3^6 \cap \Lambda^{12} \cap \Lambda^{34}.$$

The last gives an instance of $\square^2 \cdot \square^2 = 2$ in $Gr(2, 5)$. As before, letting one of $F_3^5$ or $F_3^6$ move in a pencil completes the proof in this case.

The special positions of the flags used here give the generically transverse intersections that are claimed, as shown in [23]. $\square$

1.3. **Permutation groups.** Fix a positive integer $d$. A permutation group of degree $d$ is a subgroup of $S_d, that is, it is a group $G$ together with a faithful action on $[d] := \{1, \ldots, d\}$. Write the image of $a \in [d]$ under $g \in G$ as $g(a)$. (The set $[d]$ may be replaced by any set of cardinality $d$. This permutation group is **transitive** if for all $i, j \in [d]$, there is a $g \in G$ with $g(i) = j$. More generally, for any $1 \leq t \leq d$, a permutation group $G$ is **$t$-transitive** if for any distinct $i_1, \ldots, i_t \in [d]$ and distinct $j_1, \ldots, j_t \in [d]$, there is a $g \in G$ with $g(i_m) = j_m$ for $m = 1, \ldots, t$. The full symmetric group $S_d$ is $d$-transitive and its alternating subgroup $A_d$ is $(d-2)$-transitive. These are the only highly transitive permutation groups. This is explained in [4, §4] and summarized in the following proposition, which follows from the O’Nan-Scott Theorem [22] and the classification of finite simple groups.
Theorem 4.11 of [4]. The only 6-transitive groups are the symmetric and alternating groups. The only 4-transitive groups are the symmetric and alternating groups, and the Mathieu groups $M_{11}$, $M_{12}$, $M_{23}$, and $M_{24}$. All 2-transitive permutation groups are known.

Tables 7.3 and 7.4 in [4] list the 2-transitive permutation groups. Let $G$ be a transitive permutation group of degree $d$. A block is a subset $S$ of $[d]$ such that for every $g \in G$ either $g(S) = S$ or $g(S) \cap S = \emptyset$. The orbits of a block generate a partition of $[d]$ into blocks. The group $G$ is primitive if its only blocks are $[d]$ or singletons; otherwise it is imprimitive. Any 2-transitive permutation group is primitive, and primitive permutation groups that are not symmetric or alternating are rare—the set of degrees $d$ of such nontrivial primitive permutation groups has density zero in the natural numbers [4, §4.9].

Let $[d] \times [f]$ be the set of ordered pairs $\{(a, b) \mid a \in [d], b \in [f]\}$ and $\pi : [d] \times [f] \to [f]$ the projection map. The wreath product $S_d \wr S_f$ is the symmetry group of the fibration $\pi : [d] \times [f] \to [f]$. It acts (imprimitively if $d, f > 1$) on $[d] \times [f]$, preserving the fibers of $\pi$. As an abstract group, it is the semidirect product $(S_d)^f \rtimes S_f$. Its elements are ordered pairs $((g_1, \ldots, g_f), h)$ where $g_1, \ldots, g_f \in S_d$ and $h \in S_f$, with product defined by

$$(g_1, \ldots, g_f), h)((\gamma_1, \ldots, \gamma_f), k) := ((g_1\gamma_1^{-1}(1), \ldots, g_f\gamma_f^{-1}(f)), hk).$$

Its action on $[d] \times [f]$ of ordered pairs is as follows,

$$(g_1, \ldots, g_f), h)(a, b) := (g_h(b)(a), h(b)).$$

For permutation groups $G$ and $H$ of degrees $d$ and $f$, respectively, their wreath product $G \wr H$ is the obvious subgroup of $S_d \wr S_f$. Every imprimitive permutation group is a subgroup of a wreath product of symmetric groups [4, Thm. 1.8].

The symmetric group $S_d$ is the only 2-transitive permutation group that contains a 2-cycle. Jordan gave a useful generalization.

Proposition 4 (Jordan [16]). If $G \subset S_d$ is primitive and contains a $p$-cycle for some prime number $p < d - 2$, then $G$ contains the alternating group $A_d$.

2. Lower bounds for Schubert Galois groups in $Gr(4, 9)$

We use two methods to compute lower bounds for all Schubert Galois groups in $Gr(4, 9)$. The first is due to Vakil [26] and based on his geometric Littlewood-Richardson rule [25]. When this recursive criterion holds for a Schubert problem $\lambda$, the group $\text{Gal}_\lambda$ is at least alternating. The second method computes cycle types of elements in a Galois group over $\mathbb{Q}$. Since a Galois group over $\mathbb{C}$ is a normal subgroup of the group over $\mathbb{Q}$, this is nearly a lower bound. Assuming Conjecture 2, this lower bound is the Schubert Galois group—either by a corollary to Proposition 4 the group is full symmetric, or it equals the upper bound as determined in Section 3.

2.1. Vakil’s Method. This exploits the classical method of special position in enumerative geometry to obtain information about Galois groups. Coupled with Vakil’s geometric Littlewood-Richardson rule, it provides a remarkably effective method to show that nearly all Schubert Galois groups in $Gr(4, 9)$ are at least alternating. We describe Vakil’s method, sketch the algorithm, and discuss the result of using two independent implementations to test all essential Schubert problems in $Gr(4, 9)$. 
Suppose that $\pi: X \to Y$ is a branched cover of degree $d$ with regular locus $U \subset Y$. Vakil [26, § 3] described how the monodromy action over an irreducible subvariety $Z \subset Y$ that meets $U$ affects the Galois group $\text{Gal}_\pi$. Suppose that $Z \hookrightarrow Y$ is the closed embedding of a Cartier divisor that meets $U$, where $Y$ is smooth in codimension one along $Z$. Let $W$ be the closure in $X$ of $\pi^{-1}(Z \cap U)$, and consider the fiber diagram

$$
\begin{array}{c}
W \longrightarrow X \\
p \downarrow \quad \quad \downarrow \pi \\
Z \longrightarrow Y
\end{array}
$$

where $p: W \to Z$ has degree $d$. When $W$ is irreducible or has two components the following holds, for elementary reasons.

(a) If $W$ is irreducible, then the monodromy group $\text{Gal}_p$ is a subgroup of $\text{Gal}_\pi$.

(b) If $W = W_1 \cup W_2$ with each $p_i: W_i \to Z$ a branched cover of degree $d_i$, then the monodromy group for $p$ is a subgroup of $\text{Gal}_{p_1} \times \text{Gal}_{p_2} \subset \text{Gal}_\pi$ which maps surjectively onto each factor $\text{Gal}_{p_i}$.

In the above situation, Vakil gave criteria for deducing that $\text{Gal}_\pi$ is at least alternating, based on purely group-theoretic arguments including Goursat’s Lemma.

**Vakil’s Criteria.** Suppose that we have a fiber diagram as in (5). The Galois group $\text{Gal}_\pi$ is at least alternating if one of the following holds.

(i) In Case (a), if $\text{Gal}_p$ is at least alternating.

(ii) In Case (b), if $\text{Gal}_{p_1}$ and $\text{Gal}_{p_2}$ are at least alternating and either $d_1 \neq d_2$ or $d_1 = d_2 = 1$.

In Case (b), if $d_1 = 1$ and $\text{Gal}_{p_2} = S_{d_2}$, then $\text{Gal}_\pi = S_d$.

We discuss the geometric Littlewood-Richardson rule and its consequences for studying Schubert Galois groups. The geometric Littlewood-Richardson rule is presented in Vakil’s original paper [25] and also in some detail from a different perspective in [19]. Its implications for Schubert Galois groups are explained in Section 3 of [26].

Following the description in [19], given partitions $\lambda$ and $\mu$ for $Gr(k, n)$, Vakil constructs a **checkerboard game** $\mathcal{T}_{\lambda, \mu}$, which is a tree that is a graded poset of rank $\binom{n}{2}$. The nodes of $\mathcal{T}_{\lambda, \mu}$ are labeled by certain checkerboards $\bullet$, and a node $\bullet'$ has either one child $\bullet''$ or two children, $\bullet'$ and $\bullet''$. Checkerboards labeling a leaf of $\mathcal{T}_{\lambda, \mu}$ are partitions $\nu$ that satisfy $|\nu| = |\lambda| + |\mu|$.

Let $\lambda = (\lambda^1, \ldots, \lambda^s)$ be a Schubert problem for $Gr(k, n)$. Vakil constructs a **checkerboard tournament** $\mathcal{T}_\lambda$, which is a tree composed of checkerboard games. Begin with the tree $\mathcal{T}_{\lambda^{s-1}, \lambda^s}$. A leaf of $\mathcal{T}_{\lambda^{s-1}, \lambda^s}$ is labeled by a partition $\mu$ with $|\mu| = |\lambda^{s-1}| + |\lambda^s|$. At this leaf attach the tree $\mathcal{T}_{\lambda^s, \mu}$, and do this for every leaf of $\mathcal{T}_{\lambda^{s-1}, \lambda^s}$, obtaining a new tree $\mathcal{T}'$ with two levels of checkerboard games and leaves labeled with partitions $\nu$ where $|\nu| = |\lambda^{s-2}| + |\lambda^{s-1}| + |\lambda^s|$. At each such leaf of $\mathcal{T}'$ attach the tree $\mathcal{T}_{\lambda^{s-3}, \nu}$, and continue. This constructs the checkerboard tournament $\mathcal{T}_\lambda$ which is composed of $s-1$ levels of checkerboard games, and whose leaves are labeled with the unique partition $[n-k]^k := (n-k, \ldots, n-k)$ satisfying $|(n-k)^k| = |\lambda^1| + \cdots + |\lambda^s| = k(n-k)$. 
For a node $\bullet\bullet$ of the checkerboard tournament $\mathcal{T}_\lambda$, let $\delta(\bullet\bullet)$ be the number of leaves in $\mathcal{T}_\lambda$ above $\bullet\bullet$. This satisfies the recursion that $\delta(\text{leaf}) = 1$, and for a non-leaf node $\bullet\bullet$ of $\mathcal{T}_\lambda$, we have $\delta(\bullet\bullet) = \delta(\bullet\bullet')$ when $\bullet\bullet$ has a unique child $\bullet\bullet'$ and $\delta(\bullet\bullet) = \delta(\bullet\bullet') + \delta(\bullet\bullet'')$ when $\bullet\bullet$ has two children $\bullet\bullet'$ and $\bullet\bullet''$. We summarize two consequences of the details of this construction, which are among the main results in [25, 26].

**Proposition 5.** Let $\lambda$ be a Schubert problem for $\text{Gr}(k, n)$ and construct $\mathcal{T}_\lambda$ as above.

1. If $\bullet\bullet$ is the root of $\mathcal{T}_\lambda$, then $\delta(\bullet\bullet) = d(\lambda)$.
2. If for every node $\bullet\bullet$ of $\mathcal{T}_\lambda$ with two children $\bullet\bullet'$ and $\bullet\bullet''$, either $\delta(\bullet\bullet') = 1$ or $\delta(\bullet\bullet') \neq \delta(\bullet\bullet'')$, then the Schubert Galois group $\text{Gal}_\lambda$ is at least alternating.
3. If for every node $\bullet\bullet$ of $\mathcal{T}_\lambda$ with two children $\bullet\bullet'$ and $\bullet\bullet''$, one of $\delta(\bullet\bullet')$ or $\delta(\bullet\bullet'')$ is 1, then the Schubert Galois group $\text{Gal}_\lambda$ is the full symmetric group $S_{d(\lambda)}$.

**Sketch of proof of Proposition 5.** The checkerboard game $\mathcal{T}_{\lambda,\mu}$ encodes a sequence of ‘bend-and-sometimes-break’ flat degenerations of the intersection $\Omega_\lambda F_\bullet \cap \Omega_\mu M_\bullet$ of Schubert varieties into a union of Schubert varieties $\Omega$ in a leaf of $\mathcal{T}_{\lambda,\mu}$. These follow a sequence of $\binom{n}{2}$ specializations of the pair $(F_\bullet, M_\bullet)$ of flags, starting at the root with the flags in linear general position and ending at the leaves with the flags coinciding. Let $Y_r$ be the set of pairs of flags in the $r$th special position. It is an orbit of $GL(n, \mathbb{C})$ on $\mathbb{P}^n \times \mathbb{P}^n$. Furthermore, $Y_{r+1}$ is a subset of the closure $\overline{Y_r}$ of $Y_r$ with $\overline{Y_r} \to Y_r$ the inclusion of a Cartier divisor and $\overline{Y_r}$ is smooth in codimension one along $Y_{r+1}$.

The checkerboard $\bullet\bullet$ labeling a node in $\mathcal{T}_{\lambda,\mu}$ at height $r$ encodes a position of a $k$-plane with respect to a pair of flags $(F_\bullet, M_\bullet) \in Y_r$. For $(F, M) \in Y_r$, the set $X_{\bullet\bullet}(F_\bullet, M_\bullet)$ of all such $k$-planes forms a checkerboard variety, which is irreducible. The family $X_{\bullet\bullet} \to Y_r$ whose fiber over $(F_\bullet, M_\bullet) \in Y_r$ is the checkerboard variety $X_{\bullet\bullet}(F_\bullet, M_\bullet)$ has the following property. Let $\overline{X_{\bullet\bullet}} \to \overline{Y_r}$ be its closure in $\text{Gr}(k, n) \times \mathbb{P}^n \times \mathbb{P}^n$ and $W \to Y_{r+1}$ the restriction of this closure to $Y_{r+1}$ as with (5). We have that $X_{\bullet\bullet} \cup W$ is flat over $Y_r \cup Y_{r+1}$, and that $W$ has either one or two components, depending upon whether or not $\bullet\bullet$ has one or two children in $\mathcal{T}_{\lambda,\mu}$, and these components are families $X_{\bullet\bullet'}$, where $\bullet\bullet'$ is a child of $\bullet\bullet$.

This collection of families of degenerations of $\Omega_\lambda F_\bullet \cap \Omega_\mu M_\bullet$ is the geometric Littlewood-Richardson rule. This is because if $c_{\lambda,\mu}^\nu$ is the number of leaves of $\mathcal{T}_{\lambda,\mu}$ labeled $\nu$, then the properties of these families show that

$$[\Omega_\lambda F_\bullet \cap \Omega_\mu M_\bullet] = \sum_{\nu} c_{\lambda,\mu}^\nu [\Omega_\nu F_\bullet],$$

where, for $V \subset \text{Gr}(k, n)$, $[V]$ is the cohomology class Poincaré dual to the fundamental cycle of $V$ in the homology of $\text{Gr}(k, n)$. As classes of Schubert varieties form a basis for cohomology and the class of an intersection of Schubert varieties in general position is the product of their classes, the numbers $c_{\lambda,\mu}^\nu$ are the Littlewood-Richardson numbers.

Given a Schubert problem $\lambda = (\lambda^1, \ldots, \lambda^s)$, we may splice the families in the geometric Littlewood-Richardson rule for $\mathcal{T}_{\lambda^{s-1},\lambda^s}$ into the total family $X_{\lambda} \to (\mathbb{P}^n)^s$ for $\lambda$ (4) as follows. Given a node $\bullet\bullet$ at height $r$ in $\mathcal{T}_{\lambda^{s-1},\lambda^s}$, we have a family $X_{\bullet\bullet \lambda} \to (\mathbb{P}^n)^{s-2} \times Y_r$ whose fiber over a point $(F_1^1, \ldots, F_{s-2}^1, F_\bullet, M_\bullet) \in (\mathbb{P}^n)^{s-2} \times Y_r$ is

$$\Omega_{\lambda^1} F_1^1 \cap \cdots \cap \Omega_{\lambda^{s-1}} F_{s-2}^1 \cap X_{\bullet\bullet}(F_\bullet, M_\bullet).$$
A leaf $\mu$ of $T_{s-1,\lambda}$ corresponds to the total family for the Schubert problem $(\lambda^1, \ldots, \lambda^{s-2}, \mu)$ over $(\mathbb{F}_t)^{s-1}$. As before, we may splice the tree $T_{s-2,\mu}$ of deformations into these families at the leaves of $T_{s-1,\lambda}$. Continuing in this fashion creates families at each node of the checkerboard tournament $T_{\lambda}$.

For each node $\bullet$ of $T_{\lambda}$ that is at the $r$th level in a checkerboard game $T_{s^t}\mu$, which is itself at the $t$-th stage in the construction of $T_{\lambda}$, we have a branched cover $X_{\bullet, \lambda} \rightarrow (\mathbb{F}_t)^{s-t-1} \times Y_r$. Restricting its closure to $Z = (\mathbb{F}_t)^{s-t-1} \times Y_{r+1}$, we obtain a diagram as in (5) where $W$ has one or two components, corresponding to the one or two children of $\bullet$. At a leaf of $T_{\lambda}$, the corresponding family has fiber $\Omega_{[n-k]^t} F_\bullet = \{F_\lambda\}$ over $F_\bullet$, by (2). In particular, the families at the leaves of $T_{\lambda}$ all have degree 1.

For each node $\bullet$ of $T_{\lambda}$ let $d(\bullet)$ be the degree of the family $X_{\bullet, \lambda} \rightarrow (\mathbb{F}_t)^{s-t-1} \times Y_r$. Then these degrees $d$ satisfy the same recursion over $T_{\lambda}$ and initial conditions as do the numbers $\delta$ of leaves above a given node, which proves that $\delta(\bullet)$ is the degree of the family $X_{\bullet, \lambda} \rightarrow (\mathbb{F}_t)^{s-t-1} \times Y_r$ at a node $\bullet$. Statement (1) is this observation for the root node of $T_{\lambda}$. Statements (2) and (3) follow by Vakil’s criterion, applied recursively at each node $\bullet$ of $T_{\lambda}$ with two children.

Vakil observed that Proposition 5 part (2) yields an algorithm to show that a Schubert Galois group is at least alternating. He used it for the computations reported in [26].

Algorithm 6 (Vakil’s Algorithm).

Input: A Schubert problem $\lambda$.

Output: Either “Gal$\lambda$ is at least alternating” or “Cannot determine if Gal$\lambda$ is at least alternating”.

Do: Construct $T_{\lambda}$ and recursively determine $\delta(\bullet)$ for all nodes $\bullet$ of $T_{\lambda}$. If at a node $\bullet$ with two children $\bullet'$ and $\bullet''$, if $\delta(\bullet') \neq 1$ and $\delta(\bullet') = \delta(\bullet'')$, then stop and output “Cannot determine if Gal$\lambda$ is at least alternating”.

If we have either $\delta(\bullet') = 1$ or $\delta(\bullet') \neq \delta(\bullet'')$ for all nodes $\bullet$ of $T_{\lambda}$ with two children, then stop and output “Gal$\lambda$ is at least alternating”.

Vakil implemented this in a Maple script which is available from his website. A revised version is available from the website accompanying this article.

Remark 7. The construction of the checkerboard game $T_{\lambda, \mu}$ is not symmetric in $\lambda, \mu$ so that $T_{\lambda, \mu} \neq T_{\mu, \lambda}$ in general. This is despite each having the same number of leaves with a given label $\nu$, for every $\nu$, by the Littlewood-Richardson formula. Consequently, the outcome of Vakil’s Algorithm 6 depends on the ordering of the partitions in $\lambda$. 

Table 1 summarizes the result of running Vakil’s Maple script on all Schubert problems in some small Grassmannians. For each, it records the total number of Schubert problems tested (#), the number of problems $\lambda$ for which it could not decide if Gal$\lambda$ was at least alternating (??), and the time of computation in seconds or d:h:m:s format. On each Grassmannian, all Schubert problems $\lambda$ were tested, including those with $d(\lambda) = 0$ and with $d(\lambda) = 1$, as well as all non-essential Schubert problems, except for $Gr(3, 10)$, $Gr(3, 11)$, and $Gr(4, 9)$ for which many non-essential problems were not tested.

Students Christopher Brooks and Aaron Moore worked with us to implement Vakil’s algorithm in Python. We ran the resulting software on all Schubert problems in Table 1,
Table 1. Performance of Vakil’s Maple script on different $Gr(k, n)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>#</td>
<td>1</td>
<td>4</td>
<td>10</td>
<td>23</td>
<td>47</td>
<td>90</td>
<td>164</td>
<td>288</td>
<td>488</td>
</tr>
<tr>
<td>??</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>sec</td>
<td>.01</td>
<td>.03</td>
<td>.12</td>
<td>.36</td>
<td>1.28</td>
<td>3.6</td>
<td>11.4</td>
<td>40</td>
<td>199</td>
</tr>
</tbody>
</table>

Schubert Problems in $Gr(3, n)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>#</td>
<td>4</td>
<td>39</td>
<td>270</td>
<td>1337</td>
<td>5786</td>
<td>22011</td>
<td>77305</td>
</tr>
<tr>
<td>??</td>
<td>-</td>
<td>2</td>
<td>3</td>
<td>7</td>
<td>14</td>
<td>24</td>
<td>48</td>
</tr>
<tr>
<td>d:h:m:s</td>
<td>0.29</td>
<td>5.8</td>
<td>30.2</td>
<td>3:47</td>
<td>47:5</td>
<td>7:28:18</td>
<td>11:14:55:36</td>
</tr>
</tbody>
</table>

Schubert Problems in $Gr(4, n)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>#</td>
<td>10</td>
<td>270</td>
<td>3802</td>
<td>38760</td>
</tr>
<tr>
<td>??</td>
<td>-</td>
<td>3</td>
<td>33</td>
<td>233</td>
</tr>
</tbody>
</table>

with nearly the same result. Our Python script was inconclusive for 81 of the 198,099 nontrivial essentially new Schubert problems in $Gr(3, 12)$. There was a slightly different set of Schubert problems in $Gr(4, 8)$ and $Gr(4, 9)$ for which the two implementations were unable to determine if they were at least alternating. The reason for this was mentioned in Remark 7—the two implementations construct the checkerboard tournament $T_\lambda$ differently, and this matters for those Schubert problems.

2.2. Frobenius Algorithm. This gives a lower bound on a Galois group over $\mathbb{Q}$ by computing cycle types of Frobenius elements. It exploits an asymmetry in Gröbner basis calculations—it is much faster to first reduce an ideal modulo a prime $p$ and then compute an eliminant than to first compute the eliminant and then reduce modulo $p$.

We used the Frobenius algorithm to determine, with two exceptions, the Galois group of all 448 Schubert problems in Table 1 for which Vakil’s algorithm was inconclusive. Other than 14 problems in $Gr(4, 8)$ and 149 in $Gr(4, 9)$, the remaining 284 all have Galois group the full symmetric group. The 14 in $Gr(4, 8)$ had their Galois group determined in [24, § 6], where the two exceptions were shown to have at least alternating Galois group. For each of the 149 problems in $Gr(4, 9)$, we computed sufficiently many Frobenius elements to identify a particular imprimitive permutation group as a subgroup of its Galois group. The arguments in Section 3 show that this group is equal to the Schubert Galois group, and give evidence for Conjecture 2. We now give background for the Frobenius algorithm.

Let $K$ be the splitting field of a monic irreducible univariate polynomial $f \in \mathbb{Z}[x]$ and $O_K$ be its ring of elements in $K$ that are integral over $\mathbb{Z}$. Dedekind showed that for every prime $p \in \mathbb{Z}$ not dividing the discriminant of $f$, there is an element $\sigma_p \in \text{Gal}(K/\mathbb{Q})$ in the


Galois group of $K$ over $\mathbb{Q}$ such that for every prime $\varpi$ of $\mathcal{O}_K$ above $p$ and every $z \in \mathcal{O}_K$, we have $\sigma_p(z) \equiv z^p \mod \varpi$. Thus $\sigma_p$ lifts the Frobenius map $z \mapsto z^p$ on $\mathcal{O}_K/p\mathcal{O}_K$ to $K$.

The cycle type of this Frobenius lift $\sigma_p$ is given by the degrees of the irreducible factors of $f_p := f \mod p$, as the irreducible factors give primes $\varpi$ above $p$. The condition that $p$ does not divide the discriminant of $f$ is equivalent to $f_p$ being squarefree. This gives a method to compute cycle types of elements of $\text{Gal}(K/\mathbb{Q})$. For a prime $p$, factor the reduction $f_p$, and if no factor is repeated, record the degrees of the factors. This is particularly effective due to the Chebotarev Density Theorem, which asserts that Frobenius elements are uniformly distributed for sufficiently large primes $p$.

Let $\pi: X \to Y$ be a branched cover of degree $d$ defined over $\mathbb{Q}$ with $Y$ a rational variety. For any regular value $y \in Y(\mathbb{Q})$ of $\pi$, if $K_y$ is the field of definition of all points in the fiber $\pi^{-1}(y)$, then $K_y/\mathbb{Q}$ is Galois and $\text{Gal}(K_y/\mathbb{Q})$ is a subgroup of $\text{Gal}_{\pi}(\mathbb{Q})$. By Hilbert’s Irreducibility Theorem there is a Zariski-dense open subset $U$ of $Y$ for which the Galois groups are equal for all $y \in U(\mathbb{Q})$. Ekedahl [7] showed that for a sufficiently large prime $p$, the Frobenius elements $\sigma_p(y) \in \text{Gal}(K_y/\mathbb{Q}) \subset \text{Gal}_{\pi}(\mathbb{Q})$ are uniformly distributed in $\text{Gal}_{\pi}(\mathbb{Q})$ for $y \in U(\mathbb{Q})$. Thus we may study $\text{Gal}_{\pi}(\mathbb{Q})$ by fixing a prime $p$ and computing cycle types of Frobenius elements in $\text{Gal}(K_y/\mathbb{Q})$ at points $y \in U(\mathbb{Q})$.

By Jordan’s Theorem (Proposition 4), knowing cycle types of Frobenius elements can be used to show a Galois group is full symmetric.

**Corollary 8.** Suppose that $G \subset S_d$ contains a $d$-cycle, a $(d-1)$-cycle, and an element $\sigma$ with a unique longest cycle of length a prime $p < d-2$. Then $G = S_d$.

**Proof.** As $G$ contains both a $d$- and a $(d-1)$-cycle, it is 2-transitive (hence primitive), and it is not a subgroup of the alternating group. As $\sigma^{(p-1)!}$ is a $p$-cycle (it is the inverse of the $p$-cycle in $\sigma$), Jordan’s Theorem implies that $G = S_d$. \hfill \Box

Symbolic computation, together with Frobenius lifts, Ekedahl’s Theorem, and Corollary 8 give an effective method to study Galois groups in enumerative geometry. Suppose that $f_1, \ldots, f_N \in \mathbb{C}[x_1, \ldots, x_m]$ are polynomials that generate an ideal $I$ whose variety $\mathcal{V}(I) \subset \mathbb{C}^m$ is zero-dimensional and has degree $d$. The monic generator $f(x_1)$ of the ideal $I \cap \mathbb{C}[x_1]$ is its eliminant. The following Shape Lemma (adapted from [2]) is a geometric version of the theorem of a primitive extension.

**Proposition 9 (Shape Lemma).** Suppose that $f_1, \ldots, f_N$ have integer coefficients and let $I$, $f$, and $d$ be as above. Then $f \in \mathbb{Q}[x_1]$, and after clearing denominators, we may assume that $f \in \mathbb{Z}[x_1]$. If $f$ has degree $d$ and is square-free, then the splitting field of $f$ is the field generated by the coordinates of the points in $\mathcal{V}(I) \subset \mathbb{Q}^m$.

**Remark 10.** Given polynomials $f_1, \ldots, f_N \in \mathbb{Z}[x_1, \ldots, x_m]$, Gröbner basis software (e.g. Singular [6] or Macaulay2 [10]) can compute the dimension and degree of the variety $\mathcal{V}(I) \subset \mathbb{C}^m$ of the ideal $I$ they generate, and compute eliminants. These computations take place in the ring $\mathbb{Q}[x_1, \ldots, x_m]$. The software may also reduce the polynomials modulo a prime $p$, and perform the same computations in $\mathbb{F}_p[x_1, \ldots, x_m]$ (here $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$). In prime characteristic, the computations will typically be many orders of magnitude faster. This is because the height (number of digits) of coefficients over $\mathbb{Q}$ becomes enormous, while the coefficients in computations over $\mathbb{F}_p$ have height bounded by that of $p$. 


The dimension of $\mathcal{V}(I)$ in characteristic $p$ is at least its dimension in characteristic zero. When both have dimension zero, the degree in characteristic zero, and the eliminant in $\mathbb{F}_p[x_1, \ldots, x_m]$ is the reduction modulo $p$ of the eliminant in $\mathbb{Q}[x_1, \ldots, x_m]$. Finally, the software packages compute the factorization of a univariate polynomial into irreducible factors in $\mathbb{F}_p[x_1]$. □

**Algorithm 11** (Frobenius Algorithm).

**Input**: A branched cover $\pi: X \to Y$ of degree $d \geq 8$ with $Y = \mathbb{A}^n$ and $X \subset \mathbb{A}^m \times Y$ given by a family $f_1, \ldots, f_N \in \mathbb{Z}[x_1, \ldots, x_m; y_1, \ldots, y_n]$ of integer polynomials, a positive integer $M$ and a prime number $p$.

**Output**: Either “$\text{Gal}_\pi(\mathbb{Q}) = S_d$” or a list $L$ of cycle types of Frobenius elements.

**Initialize**: Set $\text{counter} := 1$, $L := [\text{ }]$ (the empty list), and $c_d, c_{d-1}, c_{\text{prime}} := 0$.

**Do**: Choose a random $y \in \mathbb{Q}^n$ and for each $i = 1, \ldots, N$ let $g_i \in \mathbb{Z}[x_1, \ldots, x_m]$ be the result of clearing denominators and removing common divisors from the coefficients of the polynomial $f_i(x; y)$. Let $I$ be the ideal in $\mathbb{F}_p[x_1, \ldots, x_m]$ generated by the reductions modulo $p$ of $g_1, \ldots, g_N$. If $\dim(I) > 0$, then return to the start of this loop.

Otherwise, compute an eliminant $g(x_1) \in \mathbb{F}_p[x_1]$ of $I$ and its irreducible factorization

$$g(x_1) = h_1(x_1)h_2(x_1) \cdots h_s(x_1).$$

If $\deg(g) < d$ or if two factors coincide, then return to the start of this loop.

Otherwise, append to $L$ the cycle type given by the degrees of the factors.

If $s = 1$ so that $g$ is irreducible, set $c_d := 1$.

If $s = 2$ and one factor has degree $d-1$, set $c_{d-1} := 1$.

If the maximal degree of a factor is a prime between $d-2$ and $d/2$, set $c_{\text{prime}} := 1$.

If $c_d \cdot c_{d-1} \cdot c_{\text{prime}} = 1$, then stop and output “$\text{Gal}_\pi(\mathbb{Q}) = S_d$”.

If $\text{counter} = M$, then stop and output $L$.

Set $\text{counter} := \text{counter} + 1$ and return to start of this loop.

**Remark 12**. For $d \leq 7$, a more involved, but elementary, decision procedure is used to detect if $\text{Gal}_\pi(\mathbb{Q}) = S_d$. □

**Proof of correctness.** In each iteration of the Do loop, a random element $y \in \mathbb{Q}^n$ is chosen, and the algorithm tries to compute the reduction modulo $p$ of the eliminant of polynomials that define the fiber $\pi^{-1}(y)$, and then its irreducible factorization. If there is no eliminant, if it does not satisfy the Shape Lemma, or if it is not square-free, then the algorithm returns to the start of the loop, choosing another element of $\mathbb{Q}^n$.

Otherwise, the algorithm saves the degrees of the factors of the eliminant. This is the cycle type of a Frobenius element of $\text{Gal}_\pi(\mathbb{Q})$. The variables $c_d, c_{d-1},$ and $c_{\text{prime}}$, which record that a $d$-cycle, a $(d-1)$-cycle, or a permutation with a unique longest cycle of length a prime at most $d-2$ have been observed, are updated. Once each has been observed, the algorithm terminates and returns “$\text{Gal}_\pi(\mathbb{Q}) = S_d$”, which holds, by Corollary 8.

If after $M$ iterations, the three cycles from the hypothesis of Corollary 8 have not been observed, then the algorithm terminates and returns the list $L$ of observed cycle types of Frobenius elements. If there have not yet been $M$ iterations, then $\text{counter}$ is incremented and the algorithm returns to the start of the loop. The algorithm must terminate, and in either case, it returns correct output. □
Given a Schubert problem $\lambda$, the total family $X_{\lambda} \to (F_{\ell})^*$ of the Schubert problem is a branched cover defined over $\mathbb{Z}$ of degree $d(\lambda)$. As sketched at the end of Subsection 1.1, this may be formulated by polynomials with integer coefficients, and so the Frobenius algorithm may be used to study the Schubert Galois group $\text{Gal}_{\lambda}(\mathbb{Q})$.

We wrote software implementing the Frobenius algorithm to study Schubert Galois groups in small Grassmannians, particularly $Gr(4,9)$. That software, along with a more complete description and its output is found on our web page. We provide a summary.

2.2.1. $Gr(2, n)$. As all Schubert problems in $Gr(2, n)$, for any $n$, are at least alternating [3], we did not test any Schubert problems in these Grassmannians.

2.2.2. $Gr(3, n)$. Vakil’s algorithm was inconclusive for 98 Schubert problems in $Gr(3, n)$ for $n \leq 11$, as indicated in Table 1. Our Python implementation found an additional 81 inconclusive Schubert problems in $Gr(3, 12)$. Our implementation of the Frobenius algorithm showed that each of these 179 Schubert problems has $\text{Gal}_{\lambda}(\mathbb{Q}) = S_{d(\lambda)}$.

**Theorem 13.** Every Schubert problem in $Gr(3, n)$ for $n \leq 12$ has at least alternating Galois group over $\mathbb{Q}$.

The results that Schubert Galois groups in $Gr(3, n)$ are 2-transitive [24], and those for $Gr(2, n)$ are at least alternating [3], constitute evidence for the conjecture that every Schubert Galois group in $Gr(2, n)$ and in $Gr(3, n)$ is a symmetric group.

2.2.3. $Gr(4, n)$. Since $Gr(4, 6) \simeq Gr(2, 6)$ and $Gr(4, 7) \simeq Gr(3, 7)$, the next Grassmannian to study is $Gr(4, 8)$. Five of its 33 inconclusive Schubert problems from Table 1 are non-essential, and the remaining 28 were studied in [24, §6]. Two are at least alternating, and twelve more were found to be full symmetric by the Frobenius algorithm. The remaining 14 are enriched, and their Galois groups over $\mathbb{Q}$ were determined.

For $Gr(4, 9)$, our software tested each of the 233 inconclusive Schubert problems from Table 1. Of these, 79 were shown to be full symmetric. The remaining 154 appeared to be enriched. For each of these 154, we tried to compute cycle types of 50,000 Frobenius elements, which showed that the Galois group over $\mathbb{Q}$ was likely equal to a particular permutation group, either $S_2 \wr S_2$, $S_2 \wr S_3$, $S_3 \wr S_2$, $S_5 \wr S_2$, or $S_4$ acting the six equipartitions of $[4]$. Five of these, including the one with Galois group $S_4$, were not essential—they come from Schubert problems in $Gr(4, 8)$. This is also described on our web page. We tabulate how many enriched Schubert problems were found for each group.

<table>
<thead>
<tr>
<th>$S_2 \wr S_2$</th>
<th>$S_2 \wr S_3$</th>
<th>$S_3 \wr S_2$</th>
<th>$S_5 \wr S_2$</th>
<th>$S_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>99</td>
<td>20</td>
<td>25</td>
<td>9</td>
<td>1</td>
</tr>
</tbody>
</table>

We also used the Frobenius algorithm to test the 26,051 Schubert problems $\lambda$ in $Gr(4, 9)$ with $d(\lambda) \lesssim 300$; for all except the 154 enriched ones, $\text{Gal}_{\lambda}(\mathbb{Q}) = S_{d(\lambda)}$.

For two Schubert problems, $\cdot \cdot \cdot [6] \cdot [6] = 6$ and $\cdot \cdot \cdot [6] \cdot [6] \cdot [6] \cdot [6] = 6$, we computed nearly two million Frobenius elements with $p = 10007$. We display the results in Table 2. The discrepancies between two million and the number of computed Frobenius elements were instances where either the ideal $I$ in $F_{10007}[x]$ was positive-dimensional, the eliminant did not have the expected degree, or it was not square-free. Both Schubert Galois groups
Table 2. Frequency of cycle types for two Schubert problems.

<table>
<thead>
<tr>
<th>Cycle Type</th>
<th>Frequency</th>
<th>Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>(6)</td>
<td>333487</td>
<td>8.01099</td>
</tr>
<tr>
<td>(3,3)</td>
<td>332912</td>
<td>7.99718</td>
</tr>
<tr>
<td>(2,4)</td>
<td>249863</td>
<td>6.00219</td>
</tr>
<tr>
<td>(2,2,2)</td>
<td>291238</td>
<td>6.99609</td>
</tr>
<tr>
<td>(1,1,4)</td>
<td>250175</td>
<td>6.00968</td>
</tr>
<tr>
<td>(1,1,2,2)</td>
<td>373920</td>
<td>8.98227</td>
</tr>
<tr>
<td>(1,1,1,1,2)</td>
<td>125098</td>
<td>3.00509</td>
</tr>
<tr>
<td>(1,1,1,1,1)</td>
<td>41483</td>
<td>0.99650</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cycle Type</th>
<th>Frequency</th>
<th>Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>(6)</td>
<td>332926</td>
<td>11.99763</td>
</tr>
<tr>
<td>(3,3)</td>
<td>111610</td>
<td>4.02208</td>
</tr>
<tr>
<td>(2,4)</td>
<td>500352</td>
<td>18.03115</td>
</tr>
<tr>
<td>(2,2,2)</td>
<td>167264</td>
<td>6.02768</td>
</tr>
<tr>
<td>(1,2,3)</td>
<td>332565</td>
<td>11.98462</td>
</tr>
<tr>
<td>(1,1,2,2)</td>
<td>248642</td>
<td>8.96030</td>
</tr>
<tr>
<td>(1,1,1,3)</td>
<td>111434</td>
<td>4.01574</td>
</tr>
<tr>
<td>(1,1,1,2)</td>
<td>165746</td>
<td>5.97298</td>
</tr>
<tr>
<td>(1,1,1,1,1)</td>
<td>27411</td>
<td>0.98781</td>
</tr>
</tbody>
</table>

are subgroups of $S_6$. Dividing the total number of cycles computed by the number whose cycle type is $(1,1,1,1,1,1)$ (these Frobenius elements were the identity) gives 48.1686 and 72.8886, respectively. The divisors of $|S_6| = 6! = 720$ closest to these numbers are 48 and 72, and the observed cycle types are consistent with their Galois groups being $S_2 \wr S_3$ and $S_3 \wr S_2$, which have orders 48 and 72, respectively. For each cycle type, we determined the fraction (out of 48 and 72) of Frobenius elements with that cycle type. Other than the identity in the second group, the observed fraction was within 0.5% of the actual distribution in the expected Galois group. In the next section, we show that these Schubert problems have Galois group equal to $S_2 \wr S_3$ and $S_3 \wr S_2$, respectively.

3. Fibrations of Schubert Problems

The essential enriched Schubert problems in $Gr(4,9)$ all share a common structure which explains their Galois groups: their solutions form a fiber bundle with base and fibers Schubert problems in smaller Grassmannians. This is similar to the structure identified by Esterov [8] for systems of sparse polynomials and to the decomposable projections of Améndola and Rodriguez [1] that implies the corresponding Galois group is subgroup of a wreath product. Similarly, any fibered Schubert problem has Galois group a subgroup of a wreath product. In $Gr(4,9)$, there are two different ways for a Schubert problem to be fibered, and we treat each in a separate subsection.

**Definition 14.** Let $\lambda$, $\mu$, and $\nu$ be Schubert problems in $Gr(k+a, n+b)$, $Gr(k, n)$, and $Gr(a, b)$, respectively. Then $\lambda$ is fibered over $\mu$ with fiber $\nu$ if the following holds: For every general instance $F_\bullet \in (F_{k+n+b})^*$ of $\lambda$, there is a subspace $V \subset \mathbb{C}^{n+b}$ of dimension $n$ and an instance $E_\bullet$ of $\mu$ in $Gr(k, V)$ such that for every $H \in \Omega_\lambda F_\bullet$, we have $H \cap V \in \Omega_\mu E_\bullet$. Furthermore, if we set $W := \mathbb{C}^{n+b}/V$, then for any $h \in \Omega_\mu E_\bullet$, there is an instance $G_\bullet(h)$ of $\nu$ in $Gr(a, W)$ such that $H \mapsto (H \cap V, H/(H \cap V))$ is a bijection between $\Omega_\lambda F_\bullet$ and $\Omega_\mu E_\bullet$. and
\(\Omega_\mu \mathcal{E} \times \Omega_\nu \mathcal{G}(h)\). This is a fibration as the second instance \(\mathcal{G}(h)\) depends upon \(h\). Lastly, a dense set of instances of \(\mu\) in \(Gr(k, V)\) and of \(\nu\) in \(Gr(a, W)\) occur in this way. \(\diamondsuit\)

In practice, we identify a subspace \(W(h) \subset \mathbb{C}^{n+b}\) complementary to \(V\), and an instance \(\mathcal{G}(h)\) of \(\nu\) in \(Gr(a, W(h))\) such that \(H \in \Omega_\lambda \mathcal{F} \Rightarrow H = (H \cap V) \oplus (H \cap W(h))\), and

\[\Omega_\lambda \mathcal{F} = \{h \oplus K \mid h \in \Omega_\mu \mathcal{E} \text{ and } K \in \Omega_\nu \mathcal{G}(h)\}\]  

(7)

**Lemma 15.** If \(\lambda\) is a Schubert problem fibered over \(\mu\) with fiber \(\nu\), then \(d(\lambda) = d(\mu)d(\nu)\) and \(\text{Gal}_\lambda\) is a subgroup of \(\text{Gal}_\nu \wr \text{Gal}_\mu\) whose projection to \(\text{Gal}_\mu\) is surjective. Furthermore, the kernel of the surjection \(\text{Gal}_\lambda \twoheadrightarrow \text{Gal}_\mu\) is a subgroup of \(\text{Gal}_\nu^{d(\mu)}\) that is stable under the action of \(\text{Gal}_\mu\) and whose projection to each \(\text{Gal}_\nu\) factor is surjective.

**Proof.** Let \(\mathcal{F} \in (\mathbb{F}_n)^a\) be general. Then the map \(\pi: H \mapsto H \cap V\) from \(\Omega_\lambda \mathcal{F}\) to \(\Omega_\mu \mathcal{E}\) is surjective, with the fiber above \(h \in Gr(a, V)\) given by \(\Omega_\nu \mathcal{G}(h)\). Thus \(d(\lambda) = d(\mu)d(\nu)\).

The group \(\text{Gal}_\lambda\) preserves this fibration and the map \(\pi\) induces a surjective group homomorphism \(\text{Gal}_\lambda\) to \(\text{Gal}_\mu\), by the definition of fibration. Thus \(\text{Gal}_\lambda\) is a subgroup of \(\text{Gal}_\nu \wr \text{Gal}_\mu\). Furthermore, the kernel of \(\text{Gal}_\lambda \twoheadrightarrow \text{Gal}_\mu\) is stable under the action of \(\text{Gal}_\mu\) and its projection to each factor \(\text{Gal}_\nu\) is surjective, again by the definition of fibration. \(\square\)

There are two families of enriched Schubert problems in \(Gr(4, 8)\) that are fibrations, each in different way. This persists to \(Gr(4, 9)\). We treat each type of fibration in each of the next two subsections, after establishing some common notation.

For partitions \(\lambda\) and \(\mu\), let \(\lambda \triangleright \mu\) be their component-wise sum, \((\lambda \triangleright \mu)_i = \lambda_i + \mu_i\), which is always a partition. When \(\mu_{a+1} = 0\) and \(\lambda_1 = \cdots = \lambda_a = r\), so that \(\lambda\) has \(r\) columns with the last of height at least \(a\), then the columns of \(\lambda \triangleright \mu\) are the columns of \(\lambda\) concatenated with those of \(\mu\). If \(\lambda_r \geq \mu_1\) and \(\lambda_{r+1} = 0\), then \(\lambda \triangleleft \mu\) is the partition defined by concatenating the rows of \(\lambda\) and \(\mu\). Notice that if \(\lambda^t\) denotes the partition obtained from \(\lambda\) by interchanging rows with columns, then \((\lambda \triangleright \mu)^t = \lambda^t \triangleleft \mu^t\). Note that \(|\lambda \triangleright \mu| = |\lambda \triangleleft \mu| = |\lambda| + |\mu|\), whenever these operations are defined.

**Example 16.** Write \(1^a\) for the partition \((1, \ldots, 1)\) with \(a\) parts, each of size 1 and \(c\) for the partition \((c)\) with one part of size \(c\). If \(\mu = (3, 1)\), we display Young diagrams for \(1^3 \triangleright \mu\), \(4 \triangleleft \mu\), \((4 \triangleleft \mu)^t = 1^4 \triangleright (2, 1, 1)\), as well as \((2, 2) \triangleright 1\) and \((2, 2) \triangleleft 1\).

In these, we have shaded the portions \(1^3, 4, 1^4\), and \((2, 2)\). \(\diamondsuit\)

### 3.1. Fibrations of Type I

In a Schubert problem \(\nu = (\nu^1, \ldots, \nu^s)\) in \(Gr(a, b)\), some of the partitions \(\nu^i\) could be 0, and therefore impose no conditions. While these do not affect the Schubert problem, this flexibility is important to the following result.

**Theorem 17.** Suppose that \(a < b\) and \(\nu\) is a Schubert problem in \(Gr(a, b)\). Then

\[\lambda := ((b-a+1) \triangleright \nu^1, (b-a+1) \triangleleft \nu^2, 1^{a+1} \triangleright \nu^3, 1^{a+1} \triangleleft \nu^4, \nu^5, \ldots, \nu^s)\]

is a Schubert problem in \(Gr(2+a, 4+b)\) that is fibered over \(\square^4\) in \(Gr(2, 4)\) with fiber \(\nu\). Its Galois group \(\text{Gal}_\lambda\) is a subgroup of \(\text{Gal}_\nu \wr S_2\) as in Lemma 15.
Example 18. Let $a = 2$ and $b = 5$. Then $(\square^4, \square^2), (0, 0, \square^2, \square^4)$, and $(0, \square, 0, \square^4)$ are Schubert problems in $Gr(2, 5)$ that are the same geometrically, and each has five solutions. By Theorem 17 these give the following Schubert problems in $Gr(4, 9)$,

$$\square^2 \cdot \square^2 \cdot \square^2, \quad \square^2 \cdot \square^2 \cdot \square^4, \quad \text{and} \quad \square^2 \cdot \square^2 \cdot \square^4.$$

Each is fibered over $\square^4 = 2$ in $Gr(2, 4)$ with fiber $\square^6 = 5$ in $Gr(2, 5)$. By Lemma 15, each has ten solutions and its Galois group is a subgroup of $S_5 \wr S_2$.

Every nontrivial Schubert problem in $Gr(2, 5)$ gives an enriched Schubert problem in $Gr(4, 9)$ that is fibered over $\square^4 = 2$. Omitting trivial conditions and writing in multiplicative form, these Schubert problems are

$$2 = \square \cdot \square^3 = \square \cdot \square^4 = \square^2 \cdot \square^2, \quad 3 = \square \cdot \square^4, \quad \text{and} \quad 5 = \square^6,$$

and each has full symmetric Galois group, by Proposition 3. These and others geometrically equivalent to them give enriched Schubert problems whose Galois groups are subgroups of wreath products $S_d \wr S_2$, for $d = 2, 3, 5$. Our computation of Frobenius elements shows that over $\mathbb{Q}$, each is equal to the wreath product. Table 3 gives the number of Schubert problems in $Gr(4, 9)$ fibered over $\square^4 = 2$ with the given fiber.

Table 3. Fibered Schubert problems from Theorem 17.

<table>
<thead>
<tr>
<th>Schubert Problem</th>
<th>$\square \cdot \square^3$</th>
<th>$\square \cdot \square^4$</th>
<th>$\square^2 \cdot \square^2$</th>
<th>$\square \cdot \square^4$</th>
<th>$\square^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number Enriched</td>
<td>20</td>
<td>21</td>
<td>26</td>
<td>21</td>
<td>9</td>
</tr>
<tr>
<td>Galois Group</td>
<td>$S_2 \wr S_2$</td>
<td>$S_2 \wr S_2$</td>
<td>$S_2 \wr S_2$</td>
<td>$S_3 \wr S_2$</td>
<td>$S_5 \wr S_2$</td>
</tr>
</tbody>
</table>

Corollary 19. The 97 Schubert problems of Table 3 all have the claimed Galois groups.

We prove this after the proof of Theorem 17.

Remark 20. A (partial) flag $F_\bullet$ in $\mathbb{C}^n$ is a nested sequence of subspaces, where not all dimensions need occur. If all possible dimensions occur in $F_\bullet$, it is complete, as in (1).

For any subspace $h \subset \mathbb{C}^n$, the sequence of subspaces $\langle h, F_i \rangle$ for $F_i \in F_\bullet$ forms another flag $h + F_\bullet$ in $\mathbb{C}^n$ with smallest subspace $h$. If $V \subset \mathbb{C}^n$ is a subspace, then the subspaces $V \cap F_i$ form a flag $V \cap F_\bullet$ in $V$. If $\mathbb{C}^n \rightarrow V$ is surjective, then the image of $F_\bullet$ is a flag in $V$.

Proof of Theorem 17. Let $\mathcal{F}_\bullet := (F_1^\bullet, \ldots, F_s^\bullet)$ be general flags in $\mathbb{C}^{4+b}$ giving an instance of $\lambda$ with solutions $\Omega_\lambda \mathcal{F}_\bullet$. We will freely use that the flags in $\mathcal{F}_\bullet$ are in linear general position. We will show that the flags $F_1^\bullet, \ldots, F_s^\bullet$ give linear subspaces $V \cong \mathbb{C}^4$ and $W \cong \mathbb{C}^b$ in direct sum and induce flags $\mathcal{E}_\bullet := (E_1^\bullet, \ldots, E_s^\bullet)$ in $V$, giving an instance of $\square^4$ in $Gr(2, V)$ as in Definition 14. Then, for every solution $h \in \Omega_\mu \mathcal{E}_\bullet$, we construct flags $\mathcal{G}_\bullet(h)$ in $W$ that give an instance of $\nu$ in $Gr(a, W)$ and show that (7) holds for $\mu = \square^4$. The generality of $\mathcal{F}_\bullet$ implies that the flags $\mathcal{E}_\bullet$ in $V$ are general and for $h \in \Omega_\mu \mathcal{E}_\bullet$ the flags $\mathcal{G}_\bullet(h)$ in $W$ are general, which will prove the theorem.
As $\nu$ is a Schubert problem in $Gr(a, b)$, we have $a(b-a) = \sum_i |\nu^i|$. Then $\sum_i |\lambda^i|$ equals
\[
(b-a+1) + |\nu^1| + (b-a+1) + |\nu^2| + (a+1) + |\nu^3| + (a+1) + |\nu^4| + \sum_{i=5}^s |\nu^i| = 2(b-a+1) + 2(a+1) + a(b-a) = (a+2)(b+4-(a+2)),
\]
so that $\lambda$ is a Schubert problem in $Gr(2+a, 4+b)$. Define the subspaces,
\[
V := \langle F_1^1, F_2^1 \rangle \simeq \mathbb{C}^4 \quad \text{and} \quad W := F_{b+2}^3 \cap F_{b+2}^4 \simeq \mathbb{C}^b.
\]
Then $V \cap W = \{0\}$, so that $V \oplus W = \mathbb{C}^{4+b}$.

If $H \in \Omega_\lambda(F_1^1 \cap \cdots \cap \Omega_\lambda F_4^1)$, then for $i = 1, 2$ and $j = 3, 4,$
\[
\dim H \cap F_i^j \geq 1 \quad \text{and} \quad \dim H \cap F_{b+2}^j \geq a+1,
\]
as $\lambda^1$ and $\lambda^2$ both contain $(b-a+1)$ and $\lambda^3$ and $\lambda^4$ both contain $1^{a+1}$. As $V \cap W = \{0\}$, we have $\dim(H \cap V) = 2$ and $\dim(H \cap W) = a$, so that the inequalities in (8) are equalities.

Let $E_\bullet$ consist of the four flags $E_j^i := V \cap F_i^j$ for $j = 1, \ldots, 4$ induced on $V$ by $F_1^1, \ldots, F_4^1$. These give an instance of the Schubert problem $\Delta^1$ in $Gr(2, V)$, which has two solutions. For each solution $h \in \Omega^{\bullet}_\nu E_\bullet$, let $G_\bullet(h)$ be the set of flags in $W$, where for each $j = 1, \ldots, s$, $G_j^i(h) := W \cap (h+F_j^i)$ is the flag in $W$ induced by the partial flag $h+F_i$ in $\mathbb{C}^{4+b}$. This gives an instance of the Schubert problem $\nu$ in $Gr(a, W)$.

We establish (7) for $\mu = \square^4$ by showing both containments. Let $H \in \Omega_\lambda F_\bullet$. As $H \in \Omega_{\lambda^1} F_1^j$ for $j = 1, \ldots, 4$, we already showed that $h := H \cap V \in Gr(2, V)$. We claim that $h \in \Omega^{\bullet}_\nu E_\bullet$. First suppose that $j \in \{1, 2\}$. Then $E_2^j = F_2^j$. By (8) and the definitions of $h$ and $V$, $\dim h \cap E_2^j \geq 1$, so that $h \in \Omega^{\bullet}_\nu E_2^j$. Now suppose that $j \in \{3, 4\}$. Then $V \cap F_{b+2}^j = E_2^j$. As $\dim H \cap V = 2$ and $\dim H \cap F_{b+2}^j = a+1$, we have $1 \leq \dim H \cap V \cap F_{b+2}^j = \dim h \cap E_2^j$, so that $h \in \Omega^{\bullet}_\nu E_2^j$, and thus $h \in \Omega^{\bullet}_\nu E_\bullet$.

We complete the proof by showing that if $h \in \Omega^{\bullet}_\nu E_\bullet$ then for each $j = 1, \ldots, s$,
\[
K \in \Omega_{\nu} G_j^i(h) \iff h \oplus K \in \Omega_{\lambda^i} F_j^i.
\]
That is, $K$ satisfies the Schubert conditions (2) for $\nu^j$ and $G_j^i(h)$ if and only if $h \oplus K$ satisfies (2) for $\lambda^j$ and $F_j^i$. This involves careful counting and linear algebra, using that the flags $F_\bullet$ are in general position. As $h$ is fixed, write $G_j^i$ for $G_j^i(h)$.

Suppose that $j \geq 5$ and let $1 \leq c \leq b$. Since $h \cap F_{c+2}^j = \{0\}$, $\dim(h + F_{c+2}^j) = c+4$, and so $\dim(h + F_{c+2}^j) \cap W = c$. Thus $G_j^i = (h + F_{c+2}^j) \cap W$.

**Claim 1:** For $K \in Gr(a, W)$, we have that $\dim K \cap G_j^i = \dim(h \oplus K) \cap F_j^i$.

Set $d := \dim(h \oplus K) \cap (h + F_{c+2}^j)$, and observe that $h$ is a subset of the intersection. As $F_{c+2}^j$ has codimension 2 in $h + F_{c+2}^j$, this implies that $d-2 = \dim(h \oplus K) \cap F_{c+2}^j$. Similarly, as $K$ has codimension 2 in $h \oplus K$, we have $d-2 = \dim K \cap (h + F_{c+2}^j)$. Since $K \subset W$, $K \cap G_j^i = K \cap (h + F_{c+2}^j)$, which proves the claim.

To verify (9), let $i \leq a$. By Claim 1, we have
\[
\dim(K \cap G_{b-a+i}^j) = \dim(h \oplus K) \cap F_{b-a+i}^{j+2}.
\]
As $\nu^j = \lambda^j$ and $b - a + i - \nu^j_i + 2 = (b+4) - (a+2) + i - \lambda^j_i$, (2) implies that (9) holds.

We now treat the cases $j = 3, 4$. Observe first that $\dim h \cap F^3_{b+2} = 1$, as $E_2^3 = V \cap F^3_{b+2}$, and the same for $E_2^4$ and $F^4_{b+2}$. Let us investigate the flag $G^3_3$ (the same argument holds for $j = 4$). Recall that $W = F^3_{b+2} \cap F^4_{b+2}$. Suppose that $1 \leq c \leq b$. Observe that $\dim(h + F^3_{c+1}) \cap F^3_{b+2} = c+2$, and thus $\dim(h + F^3_{c+1}) \cap W = c$, as the flags are general given that $h$ meets $F^3_{b+2}$ and $F^4_{b+2}$. Thus $G^3_c = (h + F^3_{c+1}) \cap W$.

**Claim 2:** For $K \in Gr(a, W)$, we have that $\dim K \cap G^3_c = \dim(h \oplus K) \cap F^3_{c+1}$.

This follows by essentially the same arguments as Claim 1.

To show (9), let $i \leq a$. Then Claim 2 with $c = b-a+i-\nu^3_i$ implies that

$$\dim(K \cap G^3_{b-a+i-\nu^3_i}) = \dim(h \oplus K) \cap F^3_{b-a+i-\nu^3_{i+1}}.$$  

Since $\lambda^3_i = 1 + \nu^3_i$ and $b - a + i - \nu^3_i + 1 = (4b) - (2a) + i - \lambda^3_i$, we see that (2) holds for $i \leq a$. For $i = a+1$, we have $\lambda^3_{a+1} = 1$. Note that $\dim(h \oplus K) \cap F^3_{b+2} = a + 1$ as $K \subset W \subset F^3_{b+2}$ and $\dim h \cap E^3_2 = 1$, which shows that (9) holds.

We complete the proof, treating the cases $j = 1, 2$. As before, it suffices to show this for $j = 1$. Since $E_2^1 = F_2^1 \subset V$, we have $1 = \dim h \cap E_2^3 = \dim(h \oplus K) \cap F_2^1$. Let $1 \leq c \leq b$. Since $\dim h \cap F_2^1 = 1$, $\dim(h + F^1_{c+3}) = c+4$, and so $\dim(h + F^1_{c+3}) \cap W = c$. Thus $G^1_c = (h + F^1_{c+3}) \cap W$.

**Claim 3:** For $K \in Gr(a, W)$, we have that $1 + \dim K \cap G^1_i = \dim(h \oplus K) \cap F^1_{c+3}$.

Note that $K \cap G^1_c = K \cap (h + F^1_{c+3})$. Let $d := \dim(h \oplus K) \cap (h + F^1_{c+3})$. Since $F^1_{c+3}$ has codimension 1 in $h + F^1_{c+3}$, we have that $d-1 = \dim(h \oplus K) \cap F^1_{c+3}$. Similarly, as $K$ has codimension 2 in $h \oplus K$, we have that $d-2 = \dim K \cap (h + F^1_{c+3})$, which proves the claim.

We already observed that $h \oplus K$ satisfies $1 = \dim(h \oplus K) \cap F_2^1$, which is one of the conditions (2) for $\Omega^i$, $F^1_i$. For the others, note that for $i = 1, \ldots, a$, $\nu^1_i = \lambda^1_{i+1}$. Then

$$\dim(K \cap G^1_{b-a+i-\nu^1_i}) = \dim(h \oplus K) \cap F^1_{b-a+i-\nu^1_{i+3}}$$

and $b - a + i - \nu^1_i + 3 = (b+4) - (a+2) + (i+1) - \lambda^1_{i+1}$.

**Proof of Corollary 19.** Let $\lambda$ be a Schubert problem from Table 3 fibered over $\mathbb{Q}^4$ with fiber $\nu$, and $G$ be the kernel of the homomorphism $\text{Gal}_\lambda \to S_2$. Then, $\text{Gal}_\nu \simeq S_{d(\nu)}$. By Lemma 15, $G \subset S_{d(\nu)} \times S_{d(\nu)}$ is stable under the action of $S_2$ and it projects onto each $S_{d(\nu)}$ factor. We show that $G$ contains an element $\{e\} \rtimes \sigma$, with $\sigma$ a transposition. Conjugating this by elements of $G$ and by $S_2$ shows that $G$ contains all elements $\{e\} \times \sigma$ and $\sigma \times \{e\}$ for $\sigma$ a transposition, and thus that $G = (S_{d(\nu)})^2$, which will complete the proof.

Let $\pi: \mathcal{X}_\lambda \to (\mathbb{F}_{l_n})^e$ be the branched cover (4), $L$ the Galois closure of the extension $\mathbb{C}(\mathcal{X}_\lambda) / \mathbb{C}(\mathbb{F}_{l_n}^e)$, and $K$ the fixed field of $G$. Then $K$ is the quadratic extension $\mathbb{C}(\mathcal{Z}_{\sigma^4}) / \mathbb{C}(\mathbb{F}_{l_n}^e)$, where $\rho: \mathcal{Z}_{\sigma^4} \to (\mathbb{F}_{l_n})^e$ is the family of auxiliary problems $\mathbb{Q}^4$ constructed as in Theorem 17. (We are using that quadratic extensions are Galois.) Over the locus of flags in $(\mathbb{F}_{l_n})^e$ in sufficiently general position, the map $\pi$ factors through $\rho$, giving a rational map $p: \mathcal{X}_\lambda \to \mathcal{Z}_{\sigma^4}$.

We consider a subfamily of $\pi: \mathcal{X}_\lambda \to (\mathbb{F}_{l_n})^e$ over which $p$ is regular. Fix 2-planes $L_1, L_2,$ and 7-planes $\Lambda_3, \Lambda_4$ in linear general position in $\mathbb{C}^9$. Let $\mathcal{Y} \subset (\mathbb{F}_{l_n})^e$ be the space of flags
In instances of $L_3 = 4$ and $L_4 := V \cap \Lambda_4$. Every $\mathcal{F}_\bullet \in \mathcal{Y}$ gives the same auxiliary Schubert problem $\square^4$,
\[ \Omega \Omega_{1} \cap \Omega \Omega_{2} \cap \Omega \Omega_{3} \cap \Omega \Omega_{4}. \]

This has two solutions $h_1$ and $h_2$ with $h_1 \oplus h_2 = V$. The family $\mathcal{Z}_{\mathcal{F}_\bullet} \rightarrow \mathcal{Y}$ is constant with fiber $\{ h_1, h_2 \}$. The fibers $\mathcal{X}_\lambda \rightarrow \mathcal{Y}$ are solutions to the two instances of $\nu$ given by flags $G_\bullet(h_i)$ for $i = 1, 2$, which are in $Gr(2, W)$, where $W := \Lambda_3 \cap \Lambda_4 \simeq \mathbb{C}^5$.

The conditions $\square$ in instances of $\nu$ are imposed by $G_{3}^{j} := G_{3}^{j}(h_i)$ for some $j$. From the proof of Theorem 17 this is either
\begin{enumerate}
  \item (1) $(h_i + F_3^j) \cap W$ for $j \geq 5$.
  \item (2) $(h_i + F_3^j) \cap W$ for $j = 3, 4$, or
  \item (3) $(h_i + F_3^j) \cap W$ for $j = 1, 2$.
\end{enumerate}

In case (1), the map $v \mapsto (v + F_3^j) \cap W$ is an isomorphism $V \xrightarrow{\sim} W/(F_3^j \cap W)$. Since $V = h_1 \oplus h_2$, we may choose $F_3^j$ so that $G_{3}^{1j}$ and $G_{3}^{2j}$ are any two 3-planes in $W$.

In case (2), for $i = 1, 2$ and $j = 3$, $h_i \cap F_3^j = \ell_i$ is a 1-dimensional subspace with $L_3 = \ell_1 \oplus \ell_2$. As $F_3^j \cap W = F_3^j \cap F_7^4 \simeq \mathbb{C}^2$, similar arguments imply that $G_{3}^{13}$ and $G_{3}^{23}$ may be any 3-planes in $W$ that contain a common $\mathbb{C}^2$, and the same for $j = 4$. For case (3), replacing $F_3^j$ by $F_3^6 \cap F_3^7 \simeq \mathbb{C}^3$, we find the same possibilities for $G_{3}^{1j}$ and $G_{3}^{2j}$ as in (2).

From the constructions in the proof of Proposition 3, this shows that it is possible to choose flags $\mathcal{F}_\bullet \in \mathcal{Y}$ so that the first instance of $\nu$ given by $\mathcal{G}_\bullet(h_1)$ has $d(\nu)$ simple solutions, but the second instance given by $\mathcal{G}_\bullet(h_2)$ has a unique double solution. Thus $G$ has an element of the form $\{ e \} \times \sigma$ with $\sigma$ a transposition. $\square$

In addition to the five families of fibered Schubert problems in $Gr(4, 9)$ of Example 18, two more are fibered over a Schubert problem in $Gr(2, 5)$ and come from general constructions, which are similar to those of Theorem 17.

**Theorem 21.** Suppose that $0 < a < b$ and $\nu$ is a Schubert problem in $Gr(a, b)$.

(1) $\lambda = (b-a+1, b-a) \nu^1, 1^{a+1} \nu^2, 1^{a+1} \nu^3, 1^{a+1} \nu^4, \nu^5, \ldots, \nu^8)$ is a Schubert problem in $Gr(2+a, 5+b)$ fibered over $\square^4 \cdot \square^3 = 2$ in $Gr(2, 5)$ with fiber $\nu$.

(2) $\lambda = (b-a) \nu^1, (b-a+1) \nu^2, 1^{a+1} \nu^3, 1^{a+1} \nu^4, 1^{a+1} \nu^5, \nu^6, \ldots, \nu^8)$ is a Schubert problem in $Gr(2+a, 5+b)$ fibered over $\square^4 = 3$ in $Gr(2, 5)$ with fiber $\nu$.

For $a = 2$ and $b = 4$, these constructions give enriched Schubert problems in $Gr(4, 9)$ with fiber $\square^4$. The first gives eight enriched problems with Galois group $S_2 \wr S_2$, and the second gives 15 enriched problems with Galois group $S_2 \wr S_3$. The two problems below are representative of each family of enriched problems.

\[ \square^4 \cdot \square^3 \cdot \square^3 = 4 \quad \text{and} \quad \square^4 \cdot \square^4 \cdot \square^4 \cdot \square^4 = 6. \]

**Corollary 22.** The 23 enriched Schubert problems in $Gr(4, 9)$ from Theorem 21 each have Galois group as claimed.

**Proof of Theorem 21.** The proof is similar to that of Theorem 17. For each statement we show how to recover the two problems in the fibration from a general instance $\mathcal{F}_\bullet$ of $\lambda$. 

Proof of Corollary 22.\]

For (1), suppose that \( H \in \Omega_\lambda F_j^i \) for \( j = 1, \ldots, 4 \). Genericity of \( F_\bullet \) and the condition for \( j = 1 \) implies that \( \dim H \cap F_3^1 = \dim H \cap F_4^1 = 2 \), as well as \( \dim H \cap F_2^1 = 1 \). Set \( V := F_3^1 \) and \( h := H \cap V \in Gr(2, V) \) and for \( j = 1, \ldots, 4 \), let \( E_j^i := F_j^i \cap V \). Since \( E_1^i \subset \cdots \subset F_3^1 \), we have \( h \subset E_j^i \) and \( \dim h \cap E_j^i = 1 \), so that \( h \in \Omega_\mu E_j^i \). The conditions for \( j = 2, 3, 4 \) imply that \( \dim H \cap F_j^{i+3} = a+1 \). As \( V \) has codimension \( b \), \( E_j^i \cap F_j^{i+3} \cap V \) and \( \dim h \cap V = 2 \), this implies that \( \dim h \cap F_j^{i+3} = 1 \), so that \( h \in \Omega_\mu E_j^i \).

Thus \( h \in \Omega_\mu E_j^i \) where \( \mu = \square \cdot \boxtimes \).

For \( h \in \Omega_\mu E_j^i \), define

\[
W(h) := ((h + F_{b+3}^2) \cap (h + F_{b+3}^3) \cap (h + F_{b+3}^4))/h \cong \mathbb{C}^b, \tag{10}
\]

and let \( G_j^i(h) := (h + F_{b+3}^i) \cap W(h) \). Arguments similar to those for Theorem 17 complete the proof of statement (1).

For statement (2), suppose that \( H \in \Omega_\lambda F_j^i \) for \( j = 1, \ldots, 5 \). Then \( \dim H \cap F_3^1 = \dim H \cap F_4^2 = 1 \). If \( V := \langle F_3^1, F_4^2 \rangle \cong \mathbb{C}^5 \) then \( h := H \cap V \in Gr(2, V) \). If \( \mu = (\square, \square') \) and \( E_j^i := F_j^i \cap V \) for \( j = 1, \ldots, 5 \), then similar arguments as before show that \( h \in \Omega_\mu E_j^i \). For \( h \in \Omega_\mu E_j^i \), we define \( W(h) \) as in (10), but use \( F_{b+3}^i \) for \( i = 3, 4, 5 \), and set \( G_j^i(h) := (h + F_{b+3}^i) \cap W(h) \). Similar arguments as for Theorem 17 complete the proof. Note that for \( c < b \) if \( j \geq 6 \), then \( G_j^i(h) = (h + F_{c+2}^i) \cap W(h), \) if \( j = 3, 4, 5 \), then \( G_j^i(h) = (h + F_{c+2}^i) \cap W(h) \), and if \( j = 1, 2 \), then \( G_j^i(h) = (h + F_{c+4}^i) \cap W(h) \). \( \square \)

Theorems 17 and 21 are Theorems 3.9, 3.9, and 3.10 of the thesis [27].

Proof of Corollary 22. We build on ideas from the proof of Corollary 19. Start with a subset \( \mathcal{Y} \subset (\mathbb{P} \ell)_n^* \) of flags for which the auxiliary Schubert problem \( \mu \) in \( Gr(2, V) \) is constant with solutions \( \Omega_\mu E_\bullet \). The Schubert problem of the fiber above each \( h \in \Omega_\mu E_\bullet \) is a \( \square^4 \) in \( Gr(2, W(h)) \). We produce a subfamily \( \mathcal{Y}' \subset \mathcal{Y} \) of instances that induces a constant Schubert problem for all \( h \) except one, and for that one, three of the conditions are fixed while the fourth moves in a pencil. By the discussion following Figure 1 and arguments given for Corollary 19, this implies that the Galois group is as claimed.

A Schubert problem \( \lambda \) in \( Gr(4, 9) \) in Theorem 21(1) is fibered over \( \square \cdot \square^3 \) in \( Gr(2, F_5^1) \). This reduces to \( \square^4 \) in \( Gr(2, F_4^1) \). As the solutions \( h_1 \) and \( h_2 \) are independent, \( F_4^1 = h_1 \oplus h_2 \), the same arguments as in the proof of Corollary 19 lead to the family \( \mathcal{Y}' \). While \( W(h_1) \neq W(h_2) \), both are canonically isomorphic to \( \mathbb{C}^9/F_3^1 \), which allows these arguments.

Now let \( \lambda \) be a Schubert problem in Theorem 21(2) fibered over \( \square \cdot \square^4 \) with fiber \( \square^1 \). As the three solutions to \( \square \cdot \square^4 \) in \( Gr(2, 5) \) are not independent, we must modify our arguments. Let \( F_3^1, F_3^2, F_3^3, F_4^1, \) and \( F_5^1 \) be general linear subspaces of the indicated dimensions, and let \( \mathcal{Y} \subset (\mathbb{P} \ell_7)_n^* \) be the set of flags \( F_\bullet \) that are general given that those subspaces are fixed. Then if \( V := \langle F_3^1, F_3^2 \rangle \) and we set \( E_2^1 := F_2^1, E_3^2 := F_3^2, \) and \( E_3^3 := F_3^3 \cap V \) for \( j = 3, 4, 5 \), we have the auxiliary problem

\[
\Omega_{\square} E_2^1 \cap \Omega_{\square} E_3^2 \cap \Omega_{\square} E_3^3 \cap \Omega_{\square} E_3^1 \cap \Omega_{\square} E_5^1,
\]

which is constant for \( F_\bullet \in \mathcal{Y} \) and has three solutions \( h_1, h_2, h_3 \). The solution \( h_i \) meets \( E_2^1 \) in a 1-dimensional subspace \( \ell_i \) and the three solutions are independent modulo \( E_2^1 \). We
will construct $\mathcal{Y}' \subset \mathcal{Y}$ as desired by fixing all subspaces in the flags in a given general $F_\bullet \in \mathcal{Y}$ except one. We freely use the notation from the proof of Theorem 21.

There are two cases. Suppose that $\nu^j = \square$ for some $j \geq 6$. We may assume that $\nu^6 = \square$, so that $\lambda^6 = \square$, and for $h \in \{h_1, h_2, h_3\}$, $G_3^6(h) = (h + F_5^6) \cap W(h)$. The association $V \ni v \mapsto (v + F_5^6) \cap W(h)$ is a linear surjection from $V$ to $W(h)$ that depends upon $F_5^6$. In fact $F_5^6$ is its graph, and all surjections occur in this way. Set $V' := \langle h_1, h_2 \rangle \simeq \mathbb{C}^4$. Then $\ell_3 = h_3 \cap V'$. Let $F_5^6(t)$ vary in a pencil so that the resulting maps to $W(h_i)$ are constant on $V'$, but the map to $W(h_3)$ is not constant. Then the image of $h_3$ in $W(h_3)$ moves in a pencil containing the image of $\ell_3$, but the images of $h_1$ and $h_2$ are fixed in this pencil. This gives $\mathcal{Y}'$, and shows that the kernel $G$ of the map $\text{Gal}_\lambda \to S_3$ has an element of the form $\{e\} \times \{e\} \times \sigma$, for $\sigma$ a transposition, and thus $\text{Gal}_\lambda = S_2 \wr S_3$.

Suppose now that $\nu^j = 0$ for $j \geq 6$. Then $\lambda = \square^5 \cdot \square^3 \cdot \square \cdot \square$. We show this has monodromy group $S_2 \wr S_3$ by a direct symbolic computation that is documented in a Maple script that is available on our webpage. We fix flags $F_1^6, \ldots, F_4^6$ and give a pencil $F_5^6(t)$ of flags, all in $\mathbb{Q}^9$. For these flags, the auxiliary problem $\square^5 \cdot \square^4$ has three solutions $h_1, h_2, h_3 \in Gr(2, \mathbb{Q}^5)$, independent of $t$. Thus the monodromy group of this family over $\mathbb{C}$ (the coordinate $t$) is a subgroup $G$ of the kernel $\text{Gal}_\lambda \to S_3$. Note that $G \subset S_2 \times S_2 \times S_2$, where the $i$th factor is the monodromy in the $i^4$ in the fiber over $h_i$.

For each solution $h_i$, we solve the problem in the fiber over the field $\mathbb{Q}(t)$, and compute its discriminant $\delta_i(t) \in \mathbb{Q}[t]$. The local monodromy around any root of $\delta_i(t)$ is the transposition (12) in the $i$th factor of $G$. We show that these discriminants are pairwise relatively prime, which implies that $G$ contains elements $((12), e), (e, (12), e)$, and $(e, e, (12))$, and therefore $G = S_2 \times S_2 \times S_2$ and $\text{Gal}_\lambda = S_2 \wr S_3$. \hfill \square

3.2. Fibrations of Type II. The remaining 28 essential enriched Schubert problems in $Gr(4, 9)$ are fibrations in a different manner than in §3.1. Each is a special case of one of four related general constructions.

**Theorem 23.** Suppose that $a < b$ and $\nu = (\nu^1, (b-a-1) \nu^2, 1^{a-1} \nu^3, \nu^4, \ldots, \nu^s)$ is a Schubert problem in $Gr(a, b)$. Then

$$
\lambda = ((b-a+1) \nu^1, (b-a)^2 \nu^2, 2^a \nu^3, 1^{a+1} \nu^4, \nu^5, \ldots, \nu^s)
$$

is a Schubert problem in $Gr(2+a, 4+b)$ that is fibered over $\square^4$ in $Gr(2, 4)$ with fiber $\nu$.

When $a = 2$ and $b = 5$, Theorem 23 gives 18 enriched Schubert problems fibered over $\square^4 = 2$. Since $b-a-1 = 2$, the second condition in the the fiber $\nu$ is either $\square$ or $\square$ and the third is either $\square$ or $\square$. The only such $\nu$ with $d(\nu) > 1$ are (up to reordering and omitting empty conditions),

$$
\square^2 \cdot \square^2 = 2, \quad \square \cdot \square^3 = 2, \quad \text{and} \quad \square \cdot \square^4 = 3.
$$

These give eleven, four, and four enriched Schubert problems in $Gr(4, 9)$, respectively. Here is one Schubert problem in each family:

\[\begin{array}{ccc}
\square^2 \cdot \square^2 & \square \cdot \square^3 & \square \cdot \square^4 \\
\square \cdot \square^3 & \square \cdot \square^4 & \square^2 \cdot \square^2 \\
\square \cdot \square^4 & \square \cdot \square^2 & \square^3 \cdot \square^2 \\
\end{array}\]
For nearly the same reasons as Corollary 19, these have Galois groups $S_2 \wr S_2$, $S_2 \wr S_2$, and $S_3 \wr S_2$, respectively.

**Proof.** First observe that as $\nu$ is a Schubert problem for $Gr(a,b)$, we have

$$a(b-a) = |\nu| = \sum_{i=1}^{s} |\nu^i| + b-a-1 + a-1 = \sum_{i=1}^{s} |\nu^i| + b-2.$$ 

Then $|\lambda| = \sum |\nu^i| + 2b + b + 2 = a(b-a) + 2b + 4 = (a+2)(b-a+2)$, so that $\lambda$ is a Schubert problem in $Gr(a+2,b+4)$.

Let $F_\bullet = (F_1^1, \ldots, F_s^4)$ be general flags in $\mathbb{C}^{b+4}$. Let $V := \langle F_1^1, F_2^2 \rangle \cap \langle F_1^1, F_3^3 \rangle \simeq \mathbb{C}^4$, and for $i = 1, \ldots, 4$, define $E_i^i := F_i^i \cap V$. In particular $E_1^1 = F_1^1, E_2^2 = F_2^2 \cap V, E_3^3 = F_3^3 \cap V$, and $E_4^4 = F_4^4 \cap V$. These give an instance of $\boxplus^4$ in $Gr(2, V)$. From the definition (2) of a Schubert variety, we see that if $H$ lies in

$$\Omega_{(b-a-1)} F_1^1 \cap \Omega_{(b-a-2)} F_2^2 \cap \Omega_{2a} F_3^3 \cap \Omega_{1a+1} F_4^4,$$

then $H \cap V$ has dimension 2 and is a solution to $\Omega_{\boxplus^4} \mathcal{E}_\bullet$.

Set $G_2^2 := F_{b+2}^4 \cap F_2^2 \simeq \mathbb{C}^2$ and $G_{b-2}^3 := F_{b+2}^4 \cap F_3^3 \simeq \mathbb{C}^{b-2}$, and set $W := \langle G_2^2, G_{b-2}^3 \rangle \simeq \mathbb{C}^b$. If $H$ lies in (11), then $H \cap W$ has dimension $a$ and lies in $\Omega_{(b-a-1)} G_2^2 \cap \Omega_{1b-1} G_{b-2}^3$. 

Now let $h$ be a solution to $\Omega_{\boxplus^4} \mathcal{E}_\bullet$. For each $j = 1, \ldots, s$, define $G^j_4(h)$ to be $(h + F_j^4) \cap W$. Again, similar arguments as for Theorem 17 complete the proof. 

We give a related construction of enriched Schubert problems.

**Theorem 24.** Suppose that $a < b$ and $\nu = (\nu^1, (b-a-1) \mathcal{L} \nu^2, 1^{a-1} \mathcal{L} \nu^3, \nu^4, \ldots, \nu^s)$ is a Schubert problem in $Gr(a,b)$. Then

$$\lambda = (\nu^1, (b-a-1) \mathcal{L} \nu^2, 1^{a-1} \mathcal{L} \nu^3, \nu^4, \ldots, \nu^s)$$

is a Schubert problem in $Gr(2+a, 5+b)$ fibered over $\boxplus \cdot \boxplus^4$ in $Gr(2, 5)$ with fiber $\nu$.

When $a = 2$ and $b = 4$, Theorem 24 gives five Schubert problems in $Gr(4, 9)$ with fiber $\boxplus^4$ and Galois group $S_2 \wr S_3$. These problems are

$$\begin{array}{cc}
\text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} & \text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} \\
\text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} & \text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□} \cdot \text{□}
\end{array}$$

and two others. We omit the argument for the Galois group; it is essentially the same as for the family fibered over $\boxplus \cdot \boxplus^4$ given in Corollary 22.

**Proof.** We sketch the derivation of the auxiliary Schubert problem $\boxplus \cdot \boxplus^4 = 3$ and the Schubert problem $\nu$ in the fibers, and leave the verifications to the reader. Let $\mathcal{F}_\bullet \in (\mathbb{F} \ell_n)^s$ be general flags and suppose that $H \in \Omega_{\lambda} \mathcal{F}_\bullet$. Then

$$\dim H \cap F_2^1 \geq 1, \dim H \cap F_2^2 \geq 2, \dim H \cap F_3^3 \geq a, \text{ and } \dim H \cap F_{b+3}^j \geq a+1,$$

for $j = 4,5$. Let $V := \langle F_1^1, F_2^2 \rangle \cap \langle F_2^1, F_3^3 \rangle \simeq \mathbb{C}^5$ and let $E_j^i := F_j^i \cap V$ for $j = 1, \ldots, 5$. Then $H \cap V \in Gr(2, V)$ and is a solution of $\Omega_{\boxplus^4} \mathcal{E}_\bullet$.

Given $h \in \Omega_{\boxplus \cdot \boxplus^4} \mathcal{E}_\bullet$, let $W(h) \subset F_{b+3}^4 \cap F_{b+3}^5$ be spanned by

$$\langle h, F_2^2 \rangle \cap F_{b+3}^b \cap F_{b+3}^5 \simeq \mathbb{C}^2 \text{ and } \langle h, F_{b+1}^1 \rangle \cap F_{b+3}^4 \cap F_{b+3}^5 \simeq \mathbb{C}^{b-2}.$$ 

When $h \subset H$, we have that $H \cap W(h) \in \Omega_{\nu} \mathcal{G}_\bullet(h)$, where $\mathcal{G}_\bullet(h) := (h + F_\bullet^4) \cap W(h)$. 

\qed
The five remaining enriched Schubert problems on $Gr(4, 9)$ come from two additional general constructions of enriched problems.

**Theorem 25.** Suppose that $a < b$ and

$$\nu = ((b-a-1)\lambda \nu^1, (b-a-2)\lambda \nu^2, (2)^{a-1} \lambda \nu^3, \nu^4, \ldots, \nu^s)$$

is a Schubert problem in $Gr(a, b)$. Then

$$\lambda = \left((b-a-1)^2 \lambda \nu^1, (b-a-1)^2 \lambda \nu^2, ((3)^a \lambda 2) \lambda \nu^3, (1)^{a+1} \lambda \nu^4, \nu^5, \ldots, \nu^s\right)$$

is a Schubert problem in $Gr(2+a, 4+b)$ that is fibered over $\bigotimes^4$ in $Gr(2, 4)$ with fiber $\nu$.

When $a = 2$ and $b = 5$, Theorem 25 gives two enriched Schubert problems fibered over $\bigotimes^4 = 2$ with fiber $\bigotimes^2 \cdot \bigotimes^2$ and Galois group $S_2 \wr S_2$. These problems are

$$\boxed{\bigotimes \bigotimes \bigotimes \bigotimes \bigotimes} \quad \text{and} \quad \boxed{\bigotimes \bigotimes \bigotimes \bigotimes \bigotimes} \quad \square.$$

These correspond to $\nu = (\bigotimes, \bigotimes, \bigotimes, \bigotimes)$ and $\nu = (\bigotimes, \bigotimes, \bigotimes, 0, \bigotimes)$, respectively.

**Proof.** We sketch the derivation of the auxiliary Schubert problem $\bigotimes^4 = 2$ and the Schubert problem $\nu$ in the fibers, and leave the remaining verifications to the reader. Let $F_\bullet \in (\mathbb{F} \ell_\alpha)^s$ be general and suppose that $H \in \Omega_\lambda F_\bullet$. Then

$$\dim(H \cap F^1_5) \geq 2, \quad \dim(H \cap F^2_5) \geq 2, \quad \dim(H \cap F^3_{b+1}) \geq a + 1,$$

$$\dim(H \cap F^3_{b+1}) \geq a, \quad \text{and} \quad \dim(H \cap F^3_{b+2}) \geq a + 1.$$

For $i = 1, 2$, let $L^i := F^i_5 \cap F^3_{b+1}$ and set $V := \langle L^1, L^2 \rangle \simeq \mathbb{C}^4$. If we set $L^3 := F^3_{b+1} \cap V$ and $L^4 := F^4_{b+2} \cap V$, then $\dim L^i = 2$ for $i = 1, \ldots, 4$ and $H \cap V \in Gr(2, V)$ is a solution to

$$\Omega \lambda L^1 \cap \Omega \lambda L^2 \cap \Omega \lambda L^3 \cap \Omega \lambda L^4. \quad (12)$$

Let $A := F^3_{b-1} \cap F^4_{b+2}$ and $B := F^1_5 \cap F^4_{b+2}$ and set $W := \langle A, B \rangle \simeq \mathbb{C}^h$. Then for $h \subset H$ where $h$ is a solution of (12), we have that $H \cap W \in \Omega \nu G_\bullet(h)$, where $G_\bullet^1(h) := F^1_\bullet \cap W$, $G_\bullet^3(h) := F^3_\bullet \cap W$, and $G_\bullet^j(h) := (h + F^j_\bullet) \cap W$ for $j \neq 1, 3$.

We give the second construction of enriched Schubert problems.

**Theorem 26.** Suppose that $a < b$ and $\nu = ((2)^{a-1}, (b-a-1)\lambda \nu^1, \nu^2, \ldots, \nu^s)$ is a Schubert problem in $Gr(a, b)$. Then

$$\lambda = \left((b-a+1)\lambda (2)^a, (b-a-1)^2 \lambda \nu^1, (1)^{a+1} \lambda \nu^2, (1)^{a+1} \lambda \nu^3, \nu^4, \ldots, \nu^s\right)$$

is a Schubert problem in $Gr(2+a, 4+b)$ that is fibered over $\bigotimes^4$ in $Gr(2, 4)$ with fiber $\nu$.

When $a = 2$ and $b = 5$, Theorem 26 gives three enriched Schubert problems fibered over $\bigotimes^4 = 2$ with fiber $\bigotimes^2 \cdot \bigotimes^2$ and Galois group $S_2 \wr S_2$. These problems are

$$\boxed{\bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \cdot \cdot \cdot \bigotimes \bigotimes} \quad \text{and} \quad \boxed{\bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \cdot \cdot \cdot \bigotimes \bigotimes} \quad \square.$$

and the two obtained by replacing one or two occurrences of $\bigotimes \cdot \square$ with $\square$. 
Proof. We sketch the derivation of the auxiliary Schubert problem $\square^4 = 2$ and the Schubert problem $\nu$ in the fibers, and leave the verifications to the reader. Let $\mathcal{F}_\bullet \in (\mathbb{F}_\ell)_s$ be general and suppose that $H \in \Omega_\chi \mathcal{F}_\bullet$. Then
\[
\dim(H \cap F_2^1) \geq 1, \quad \dim(H \cap F_{b+1}^1) \geq a + 1, \quad \dim(H \cap F_5^2) \geq 2, \\
\dim(H \cap F_{b+2}^3) \geq a + 1, \quad \text{and} \quad \dim(H \cap F_{b+4}^4) \geq a + 1.
\]
Let $L^1 := F_2^1$ and $L^2 := F_5^2 \cap F_{b+1}^1$ and set $V := \langle L^1, L^2 \rangle \simeq \mathbb{C}^4$. If we set $L^j := F_{b+2}^j \cap V$ for $j = 3, 4$, then $\dim(L^j) = 2$ and $H \cap V$ lies in
\[
\Omega_\nu L^1 \cap \Omega_\nu L^2 \cap \Omega_\nu L^3 \cap \Omega_\nu L^4. \tag{13}
\]
Set $W := F_{b+2}^3 \cap F_{b+4}^4 \simeq \mathbb{C}^b$ and $G_\bullet^2 := F_\bullet \cap W$. For $h$ in the intersection (13), let $G_j^2(h) := (h + F_\bullet^j) \cap W$ for $j = 2, \ldots, s$. Then have that $H \cap W(h) \in \Omega_\nu G_\bullet^2(h)$. \hfill $\square$

4. Conclusion

We studied the Galois groups of all 81,533 nontrivial Schubert problems in $Gr(4,9)$. Of the 31,806 essential problems, 149 had Galois group that did not contain the alternating group, and we identified the Galois group of these 149 enriched problems. We discussed several methods to study Schubert Galois groups, including Vakil’s algorithm and computing Frobenius elements. We also introduced a new structure, a fibration of Schubert problems, that explained the Galois groups of the 149 enriched problems. This is both a first step towards the inverse Galois problem in Schubert calculus and points towards a potential classification of enriched Schubert problems.

References

10. D. R. Grayson and M. E. Stillman, Macaulay2, a software system for research in algebraic geometry, Available at http://www.math.uiuc.edu/Macaulay2/.

Abraham Martín del Campo, Centro de Investigación en Matemáticas, A.C., Jalisco S/N, Col. Valenciana, 36023 Guanajuato, Gto. México
E-mail address: abraham.mc@cimat.mx
URL: http://personal.cimat.mx:8181/~abraham.mc

Frank Sottile, Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA
E-mail address: sottile@math.tamu.edu
URL: www.math.tamu.edu/~sottile

Robert Lee Williams, Department of Mathematics, Rose-Hulman Institute of Technology, Terre Haute, IN 47803, USA
E-mail address: william7@rose-hulman.edu
URL: www.rose-hulman.edu/academics/faculty/williams-robert-william7.html