Semialgebraic Splines

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Work with Michael DiPasquale
Motivating Goals

Compute dimensions of spline spaces on a complex of semialgebraic cells, to illustrate some phenomena not observed in traditional splines on simplicial or polyhedral complexes, and also to compare to traditional splines.

Definition: A (basic) semialgebraic set is one of the form
\[ \{ x \in \mathbb{R}^2 \mid h_i(x) \geq 0, \text{ for } i = 1, \ldots, m \} , \]
where \( h_1, \ldots, h_m \) are polynomials.
Semialgebraic Splines

A *semialgebraic spline* is a function that is piecewise a polynomial with respect to a complex $\Delta$ whose cells are semialgebraic sets.

$C^r_d(\Delta)$ : vector space of splines on $\Delta$ of degree $\leq d$ and smoothness $r$.

Let $\Delta$ be a planar complex with edges defined by real polynomials. We previously showed the homological approach of Billera-Rose-Schenck-Stillman computes the spline module $C^r_d(\Delta)$ and thus $\dim C^r_d(\Delta)$.

When $\Delta$ has a single interior vertex, $v$, we determined $\dim C^r_d(\Delta)$ in two extreme cases:

- The curves incident to $v$ form a *pencil* (as lines incident to $v$ do)
- The curves incident to $v$ have distinct tangents at $v$ and are sufficiently *generic* (a classical case for rectilinear splines).

We continue these two cases for more involved complexes $\Delta$. 

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Nets

Suppose that the edge forms span a two-dimensional space of forms (a net). Then the forms at a vertex form a pencil, and if the vertices are in general position, there is a unique edge between two vertices.

At right is the net spanned by \( \{x^2 - yz, y^2 - xz, z^2 + xy\} \) with the indicated vertices.

A net defines a map \( \varphi : \mathbb{P}^2 \to \mathbb{P}^2 \), and \( \varphi(\Delta) \) is a rectilinear complex on \( \mathbb{R}^2 \).

The spline module for \( \Delta \) is a flat base-change along \( \varphi^* \) of the spline module for \( \varphi(\Delta) \).

This gives simple formulas for \( \dim C^r_d(\Delta) \) in terms of \( \dim C^r_j(\varphi(\Delta)) \).
In the example from the previous page, here are $\Delta$ and $\varphi(\Delta)$:

$$
\begin{align*}
\dim C^1_1(d(\Delta')) &= 1, 3, 6, 10, 15, 21, 34, 54, 81, 115 \\
\dim C^1_1(d(\Delta)) &= 1, 3, 6, 10, 16, 24, 37, 55, 81, 115
\end{align*}
$$

The difference 1,3,3,1 is $\dim \mathbb{R}[x, y, z]/\langle x^2 - yz, y^2 - xz, z^2 + xy \rangle$.
In the example from the previous page, here are $\Delta$ and $\varphi(\Delta)$:

These exhibit the Morgan-Scott phenomena. Let $\Delta'$ be a generic complex from this net with the same topology as $\Delta$. Then we have

\[
\begin{array}{cccccccccc}
  d & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
  \dim C^1_d(\Delta') & 1 & 3 & 6 & 10 & 15 & 21 & 34 & 54 & 81 & 115 \\
  \dim C^1_d(\Delta) & 1 & 3 & 6 & 10 & 16 & 24 & 37 & 55 & 81 & 115 \\
\end{array}
\]

The difference $1, 3, 3, 1$ is $\dim \mathbb{R}[x, y, z]/\langle x^2 - yz, y^2 - xz, z^2 + xy \rangle$
When $\Delta$ has a single interior vertex $v$ and the edge forms at $v$ have distinct tangents, we gave a combinatorial formula for the dimension of $C^r_d(\Delta)$ for $d$ sufficiently large.

For more general $\Delta$, a more subtle acyclicity condition from local cohomology, which appeared in work of Schenck and Stillman, also gives a combinatorial formula for $\dim C^r_d(\Delta)$ for $d$ sufficiently large.
What is the point of Generic?

Formulas for $\dim C^r_d(\Delta)$ for $d$ large have two pieces:

- A regular, combinatorial part, and
- A possible difficult homology module.

For a generic complex, the regular part is even more regular, (this is a consequence of our earlier paper), and the difficult homology module has finite length, so in the long run it does not contribute, and we get a formula for $\dim C^r_d(\Delta)$ for $d$ large:

$$(\phi_2 - \phi_1) \left( \frac{d+2}{2} \right) + \sum_{\tau \in \Delta_1^0} \left( \frac{d-(r+1)n_\tau+2}{2} \right) + \sum_{v \in \Delta_0^0} \left( \frac{(r+a_v+2)}{2} - t_v \left( \frac{a_v+1}{2} \right) \right)$$

Here, $\phi_2, \phi_1, n_\tau, a_v, t_v$ are combinatorial data from the complex $\Delta$.

The point of this work is not the formulae, but rather that methods for splines on rectilinear complexes mostly also work for semialgebraic complexes.
References


