Solving Sparse Decomposable Systems

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Work with Taylor Brysiewicz, Jose Rodriguez, and Thomas Yahl
Solving Structured Systems

**Goal:** Develop numerical methods to solve systems of equations that exploit natural structures of the equations.

**My current favourite structure:**
A family of systems of equations $F(x) = 0$ on $\mathbb{C}^n \ (x \in \mathbb{C}^n)$ parameterized by $\mathbb{C}^N \ (F \in \mathbb{C}^N)$ has an incidence variety

$$\mathcal{X} := \{(x, F) \in \mathbb{C}^n \times \mathbb{C}^N | F(x) = 0\}.$$

The projection $\pi : \mathcal{X} \rightarrow \mathbb{C}^N$ has fibre $\pi^{-1}(F) = \{x \mid F(x) = 0\}$.

This is a *branched cover*, over an open set $U \subset \mathbb{C}^N$, $\mathcal{X}|_U \rightarrow U$ is a covering space with monodromy group $G_{\pi}$, the *Galois group* of this family $\pi : \mathcal{X} \rightarrow \mathbb{C}^N$ of equations. ($G_{\pi}$ is a Galois group in the usual sense.)
Recall: $\pi: \mathcal{X} \rightarrow \mathbb{C}^N$ with $\pi^{-1}(F) = \{x \mid F(x) = 0\}$.

The monodromy group $G_\pi$ of this branched cover acts on fibers. This action is imprimitive if $G_\pi$ preserves a nontrivial partition.

**Example.** The dihedral group $D_6$ acts imprimitively on the vertices of the hexagon, preserving opposite pairs of vertices. This gives: $\mathbb{Z}/2\mathbb{Z} \hookrightarrow D_6 \twoheadrightarrow S_3$.

**Proposition.** $G_\pi$ is imprimitive if and only if $\pi$ factors $\pi: \mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathbb{C}^N$ as a composition of nontrivial branched covers.

Améndola and Rodriguez explained how to exploit such a decomposable branched cover ($\ast$) in numerical algebraic geometry.

**Obstruction:** How to compute such a decomposition.
Sparse Polynomial Systems

A point \( a \in \mathbb{Z}^n \) corresponds to a monomial \( x^a := x_1^{a_1} \cdots x_n^{a_n} \).

Let \( \mathcal{A} \subset \mathbb{Z}^n \) be finite with \( 0 \in \mathcal{A} \). Then \( f = \sum_{a \in \mathcal{A}} c_a x^a \) for \( c_a \in \mathbb{C} \) is a \textit{sparse polynomial} with \textit{support} \( \mathcal{A} \). Write \( f \in \mathbb{C}^\mathcal{A} \).

Example. The support of
\[
\begin{align*}
f &= 1 + 2x^3y \\
&+ 3x^6y^2 + 4xy^3 + 5x^4y^4 + 6x^7y^5 \\
&+ 7x^2y^6 + 8x^5y^7
\end{align*}
\]
is at right.

Let \( \mathcal{A}_\bullet = \mathcal{A}_1, \ldots, \mathcal{A}_n \) with \( 0 \in \mathcal{A}_i \subset \mathbb{Z}^n \).
\[
F = (f_1, \ldots, f_n) \in \mathbb{C}^\mathcal{A}_\bullet = \mathbb{C}^{\mathcal{A}_1} \times \cdots \times \mathbb{C}^{\mathcal{A}_n}
\]
is a \textit{system of polynomials with support} \( \mathcal{A}_\bullet \).

Theorem. \textit{(Kushnirenko-Bernstein)} The number of solutions in \( (\mathbb{C}^\times)^n \) to a general system with support \( \mathcal{A}_\bullet \) is the mixed volume \( \text{MV}(\mathcal{A}_\bullet) \) of the convex hulls of the \( \mathcal{A}_i \).
Esterov’s Theorem

As before, the incidence variety
\[ \mathcal{X}_{A^\bullet} := \{ (x, F) \in (\mathbb{C}^\times)^n \times \mathbb{C}^{A^\bullet} | F(x) = 0 \} \]
is a branched cover over \( \mathbb{C}^{A^\bullet} \) with Galois group \( G_{A^\bullet} \).

This has two sources of imprimitivity

(1) Lacunary. For example, \( f(x) = g(x^3) \).
(2) Triangular. For example, \( f(x, y) = g(x) = 0 \).

For both, the solutions of \( f \) given a solution of \( g \) are the preserved partition.

**Theorem.** (Esterov) \( G_{A^\bullet} \) is the symmetric group if neither (1) nor (2) occurs. Otherwise, \( G_{A^\bullet} \) is imprimitive (besides trivial cases).

We now explain these two cases of lacunary and triangular supports.
Suppose that $\mathcal{A}_\bullet = \mathcal{A}_1, \ldots, \mathcal{A}_n$ are supports with $0 \in \mathcal{A}_i$, and the span $\mathbb{Z}\mathcal{A}_\bullet \subset \mathbb{Z}^n$ has rank $n$.

Smith normal form of the matrix whose columns are $\mathcal{A}_\bullet \sim d_1, \ldots, d_n \in \mathbb{N}$ and coordinate changes such that $\mathcal{A}_i \subset d_1\mathbb{Z} \oplus d_2\mathbb{Z} \oplus \cdots \oplus d_n\mathbb{Z}$.

Then $f_i(x) = g_i(x_1^{d_1}, \ldots, x_n^{d_n})$, where support of $g_i$ is $\mathcal{B}_i = \text{diag}(\frac{1}{d_1}, \ldots, \frac{1}{d_n})\mathcal{A}_i$.

To solve $F = 0$:

1. Solve $g_1 = \cdots = g_n = 0$.
2. For each solution $y$, get solutions $x$ of $F$ with coordinates

$$x_j := \exp\left(\frac{2\pi \arg(y_j)\sqrt{-1}}{d_j}\right)|y_j|^{\frac{1}{d_j}},$$

up to $d_j$-th roots of unity.

The Galois group is imprimitive if $MV(\mathcal{B}_\bullet) > 1$ and $d_1 \cdots d_n > 1$. 

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Triangular

After permuting and changing coordinates using Smith normal form, 
\( \mathbb{Z}\{A_1, \ldots, A_k\} \subset \mathbb{Z}^k \oplus 0^{n-k} \) and has rank \( k \).

This gives a projection \( p : \mathbb{Z}^k \oplus \mathbb{Z}^{n-k} \rightarrow \mathbb{Z}^{n-k} \) and 
corresponding coordinates \( (x, z) \in (\mathbb{C}^\times)^k \times (\mathbb{C}^\times)^{n-k} \).

To solve \( F = 0 \): 
(1) Solve \( f_1(x) = \cdots = f_k(x) = 0 \) in \( (\mathbb{C}^\times)^k \).
(2) For each solution \( y \), solve the new system

\[
G : f_{k+1}(y, z) = \cdots = f_n(y, z) = 0,
\]

which has support \( p(A_{k+1}), \ldots, p(A_n) \).

The Galois group is imprimitive when \( 1 \leq k < n \) and 
\( MV(A_1, \ldots, A_k) > 1 \) and \( MV(p(A_{k+1}), \ldots, p(A_n)) > 1 \).
(Recursive) Algorithm

Given a polynomial system $F$ with support $\mathcal{A}_\bullet$,

If neither lacunary nor triangular, call PHCpack to solve, otherwise:

If lacunary follow the algorithm given two pages ago.

If triangular follow the algorithm given on last page.

On (admittedly) manufactured examples of systems that are lacunary and/or triangular, perhaps with several levels of structure, this algorithm outperforms PHCpack.

Moral: Exploit structure. Understand Galois groups.

Thanks! Paper to come.....