Critical Points of Discrete Periodic Operators

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Critical Points of Discrete Periodic Operators

It is a widely held belief in physics that the dispersion relation of a sufficiently general periodic operator is non-degenerate near the edges of its spectrum. This is needed, for example, for effective masses in solid state physics, and many other properties (ask Peter Kuchment).

This belief holds if the critical points of the function $\lambda$ on the dispersion relation are Morse non-degenerate.

For a discrete operator on a periodic graph, the dispersion relation is an algebraic variety and degeneracy is an algebraic condition. Thus (non)degeneracy may be studied through the lens of algebraic geometry. Today, I discuss some early results in this program.
Discrete Periodic Laplace-Beltrami Operators

Let $\Gamma \subset \mathbb{R}^n$ be a $\mathbb{Z}^n$-periodic graph with periodic labels $c_e \in \mathbb{C}$ for each edge $e$. The discrete Laplace-Beltrami operator $L = L_c$ is

$$L_c f(u) := \sum c_e (f(u) - f(v)),$$

the sum over edges $e = (u, v)$ of $\Gamma$.

We consider the equation

$$L_c f = \lambda f$$

for $\lambda \in \mathbb{C}$.

Floquet theory seeks quasi-periodic solutions $f$ for $z \in (\mathbb{C}^\times)^n$:

$$f(v + \alpha) = z^\alpha f(v)$$

for $\alpha \in \mathbb{Z}^n$.

Such a function $f$ is determined by its values at the vertices of a fundamental domain $G$, so that $f \in \mathbb{C}^{V(G)} = \mathbb{C}^d$.

In the example, $n = 2$, $d = 2$ ($G = K_2$ is diatomic), and $c \in \mathbb{C}^9$. 

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\[ L_c(z)f(u) = c_1(f(u) - z_2^{-1}f(u)) \]
\[ + c_1(f(u) - z_2f(u)) \]
\[ + c_2(f(u) - z_2f(v)) \]
\[ + c_3(f(u) - z_1f(v)) \]
\[ + c_4(f(u) - z_1^{-1}f(v)) \]
\[ + c_4(f(u) - z_1f(v)) \]
\[ + c_5(f(u) - f(v)) \]
\[ + c_6(f(u) - z_2^{-1}f(v)) \]
\[ + c_9(f(u) - z_1^{-1}f(v)). \]

\[ L_c(z)f(u) = f(u)(2c_1 + c_2 + c_3 + 2c_4 + c_5 + c_6 + c_9 \]
\[ - c_1(z_2 + z_2^{-1}) - c_4(z_1 + z_1^{-1})) \]
\[ - f(v)(c_2z_2 + c_3z_1 + c_5 + c_6z_2^{-1} + c_9z_1^{-1}). \]
Set $d = \#V(G)$ and let $N$ be the number of orbits of edges in $\Gamma$. 

The **dispersion relation** $D = D_c$ is

$$\{(z, \lambda) \mid \exists f \in \mathbb{C}^d \ L_c(z)f = \lambda f\},$$

which is defined in $(\mathbb{C}_z^\times)^n \times \mathbb{C}_\lambda$ by

$$\Phi := \det(L_c(z) - \lambda I_d) = 0.$$ 

Set $U_{dg} := \{c \in \mathbb{C}^N \mid D_c \text{ has a degenerate } \lambda\text{-critical point}\}$

**Theorem.** (DKS) *Exactly one of $U_{dg}$ or $\mathbb{C}^N \setminus U_{dg}$ is dense in $\mathbb{C}^N$, and the other lies in a proper algebraic hypersurface.*

This is because the set of points $(c, p)$ with $p \in D_c$ a degenerate critical point for the function $\lambda$ is an algebraic subset of $\mathbb{C}^N \times (\mathbb{C}^\times)^n \times \mathbb{C}$. 
A **dense** periodic graph $\Gamma$ is one with as many edges as possible.

(1) The fundamental domain $G$ of $\Gamma$ is a complete graph $K_d$.

(2) If there is an edge between $G$ and $G + \alpha$ for some $\alpha \in \mathbb{Z}^n$, then $\Gamma$ has all possible edges between $G$ and $G + \alpha$.

Let $\mathcal{A} := \{ \alpha \in \mathbb{Z}^n \mid \exists \text{ an edge between } G \text{ and } G + \alpha \}$. Then for $u \in V(G)$, 

$$L_c(z) f(u) = \sum_{v \in V(G)} g_{u,v}(c, z) f(v),$$

where the matrix entry $g_{u,v}(c, z)$ is a Laurent polynomial in $z$ whose coefficients are linear in $c$ and its monomials are $\{ z^\alpha \mid \alpha \in \mathcal{A} \}$.

$\Gamma = \includegraphics[width=0.3\textwidth]{diagram1.png}$ \hspace{1cm} $\mathcal{A} = \includegraphics[width=0.1\textwidth]{diagram2.png}$ \hspace{1cm} $\left( \mathcal{A} = -\mathcal{A} \text{ is always symmetric} \right)$
Critical Point Equations

$L_c(z)$ is a $d \times d$ matrix of Laurent polynomials, each with support $\mathcal{A}$.

$\Phi = \det(L_c(z) - \lambda I_d)$ is a Laurent polynomial with support $d \cdot (\mathcal{A} \cup \{(0^n, 1)\})$, a pyramid $P$ (the Newton polytope of $\Phi$).

Critical points are defined by the equations $0 = \frac{\partial \lambda}{\partial z_i}, \ i = 1, \ldots, n$.

As $0 = \frac{\partial \Phi}{\partial z_i} + \frac{\partial \Phi}{\partial \lambda} \frac{\partial \lambda}{\partial z_i}$ and $z_i \neq 0$, the Critical Point Equations are

$$ \Phi, \ z_i \frac{\partial \Phi}{\partial z_i}, \ i = 1, \ldots, n.$$ 

Let $P_i$ be the Newton polytope of $z_i \frac{\partial \Phi}{\partial z_i}$.

We have the following estimate on the number of critical points.

**Bernstein’s Theorem.** For $c \in \mathbb{C}^N$, the number of $\lambda$-critical points on $D_c$ is bounded above by the mixed volume of $P, P_1, \ldots, P_n$. 

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In our running example

\[ \Gamma = \quad \mathcal{A} = \]

The Newton polytopes of \( \Phi, z_1 \frac{\partial \Phi}{\partial z_1}, \) and \( z_2 \frac{\partial \Phi}{\partial z_2} \) are

\[ P \quad P_1 \quad P_2 \]

Bernstein’s Theorem. For \( c \in \mathbb{C}^9, \) the number of critical points on \( D_c \) is bounded above by the mixed volume of \( P, P_1, P_2. \)
Monotonicity

Since $P_i \subset P$, we have $MV(P, P_1, \ldots, P_n) \leq (n+1)! \cdot \text{vol}(P)$.

**Theorem. (Rojas)** We have equality if for every $k$-face $F$ of $P$, $k$ of the $P_i$ have a $k$-face along $F$.

This holds in our example,

\[
\begin{align*}
P & \quad P_1 & \quad P_2
\end{align*}
\]

**Theorem. (Faust-S.)** For any dense graph, $MV = (n+1)! \cdot \text{vol}(P)$.

The key to Rojas’s result is a lemma of Bernstein:

**Bernstein’s Lemma.** The number of solutions to the critical point equations is less than the mixed volume if and only if there is a face $F$ of $P$ such that the critical point equations restricted to $F$ have a solution.
Toric Varieties

For $c \in \mathbb{C}^N$, $D_c = \{ \Phi(c, z) = 0 \} \subset (\mathbb{C}^\times)^n \times \mathbb{C}$.

Its closure is a hypersurface in the toric variety $X_P$ corresponding to $P$.

This has the property that $X_P \supset (\mathbb{C}^\times)^n \times \mathbb{C}$, and the difference $X_P \setminus (\mathbb{C}^\times)^n \times \mathbb{C}$ is a union of toric varieties $X_F$ for a face $F$ of $P$.

Also, $\overline{D_c} \cap X_F$ is defined by the restriction of $\Phi(c, z)$ to $F$.

Lemma. (Faust-S.) If $\overline{D_c} \cap X_F$ is smooth for all faces $F$ of $P$, then the number of critical points is $(n+1)! \cdot \text{vol}(P)$.

In our example, we have $3! \cdot \text{vol}(\triangle) = 32$. 

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