Chapter 1

Varieties

Algebraic geometry uses tools from algebra to study geometric sets called varieties, which are the common zeroes of a collection of polynomials. We develop some basic notions of algebraic geometry, perhaps the most fundamental being the dictionary between algebraic and geometric concepts. The basic objects we introduce and concepts we develop will be used throughout the book. These incude affine varieties, important notions from the algebra-geometry dictionary, and projective varieties. We provide additional algebraic background in the appendices and pointers to other sources of introductions to algebraic geometry in the references provided at the end of the chapter.

1.1 Affine Varieties

Let $\mathbb{K}$ be a field, which for us will almost always be either the complex numbers $\mathbb{C}$, the real numbers $\mathbb{R}$, or the rational numbers $\mathbb{Q}$. These different fields have their individual strengths and weaknesses. The complex numbers are *algebraically closed*; every univariate polynomial has a complex root. Algebraic geometry works best when using an algebraically closed field, and most introductory texts restrict themselves to the complex numbers. However, quite often real number answers are needed in applications. Because of this, we will often consider real varieties and work over $\mathbb{R}$. Symbolic computation provides many useful tools for algebraic geometry, but it requires a field such as $\mathbb{Q}$, which can be represented on a computer. Much of what we do remains true for arbitrary fields, such as the Gaussian rationals $\mathbb{Q}[i]$, or $\mathbb{C}(t)$, the field of rational functions in the variable t, or finite fields. We will at times use this added generality.

Algebraic geometry is fundamentally about the interplay of algebra and geometry, with its most basic objects the ring $\mathbb{K}[x_1, \ldots, x_n]$ of polynomials in indeterminates $x_1, \ldots, x_n$ with coefficients in $\mathbb{K}$, and the space $\mathbb{K}^n$ of n-tuples $a = (a_1, \ldots, a_n)$ of numbers from $\mathbb{K}$. We regard $\mathbb{K}^n$ as the domain of polynomials in $\mathbb{K}[x_1, \ldots, x_n]$, which are then functions from $\mathbb{K}^n \to \mathbb{K}$. We make our main definition.

Definition 1.1.1. An *affine variety* is the set of common zeroes of a collection of polynomials. Given a set $S \subset \mathbb{K}[x_1, \ldots, x_n]$ of polynomials, the affine variety defined by S is the set

$$\mathcal{V}(S) \ := \ \{a \in \mathbb{K}^n \mid f(a) = 0 \quad \text{for } f \in S\}\,.$$

This is a(n *affine*) *subvariety* of $\mathbb{K}^n$ or simply a *variety* or *algebraic variety*.

If X and Y are varieties with $Y \subset X$, then Y is a *subvariety* of X. In Exercise 2, you will be asked to show that if $S \subset T$, then $\mathcal{V}(S) \supset \mathcal{V}(T)$.

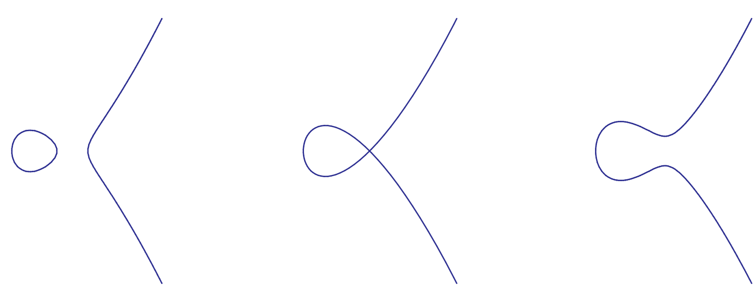
The empty set $\emptyset = \mathcal{V}(1)$ and affine space itself $\mathbb{K}^n = \mathcal{V}(0)$ are varieties. Any linear or affine subspace L of $\mathbb{K}^n$ is a variety. Indeed, an affine subspace L has an equation $Ax = b$, where A is a matrix and b is a vector, and so $L = \mathcal{V}(Ax - b)$ is defined by the linear polynomials which form the rows of the column vector $Ax - b$. An important special case is when $L = \{b\}$ is a point of $\mathbb{K}^n$. Writing $b = (b_1, \ldots, b_n)$, then L is defined by the equations $x_i - b_i = 0$ for $i = 1, \ldots, n$.

Any finite subset $Z \subset \mathbb{K}^1$ is a variety as $Z = \mathcal{V}(f)$, where

$$f \ := \ \prod_{z \in Z} (x - z)$$

is the monic polynomial with simple zeroes in Z.

A non-constant polynomial $f(x, y)$ in the variables x and y defines a *plane curve* $\mathcal{V}(f) \subset \mathbb{K}^2$. Here are the plane cubic curves $\mathcal{V}(f + \frac{1}{20})$, $\mathcal{V}(f)$, and $\mathcal{V}(f - \frac{1}{20})$, where $f(x, y) := y^2 - x^3 - x^2$.



A *quadric* is a variety defined by a single quadratic polynomial. The smooth quadrics in $\mathbb{K}^2$ are the plane conics (circles, ellipses, parabolas, and hyperbolas in $\mathbb{R}^2$) and the smooth quadrics in $\mathbb{R}^3$ are the spheres, ellipsoids, paraboloids, and hyperboloids. Figure 1.1 shows a hyperbolic paraboloid $\mathcal{V}(xy + z)$ and a hyperboloid of one sheet $\mathcal{V}(x^2 - x + y^2 + yz)$.

These examples, finite subsets of $\mathbb{K}^1$, plane curves, and quadrics, are varieties defined by a single polynomial and are called *hypersurfaces*. Any variety is an intersection of hypersurfaces, one for each polynomial defining the variety.

The set of four points $\{(-2, -1), (-1, 1), (1, -1), (1, 2)\}$ in $\mathbb{K}^2$ is a variety. It is the

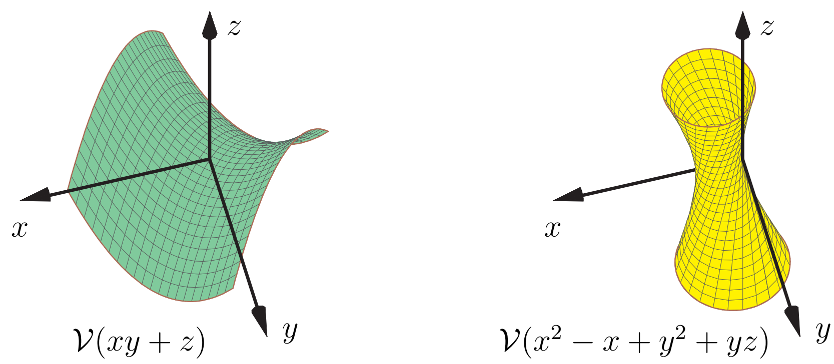
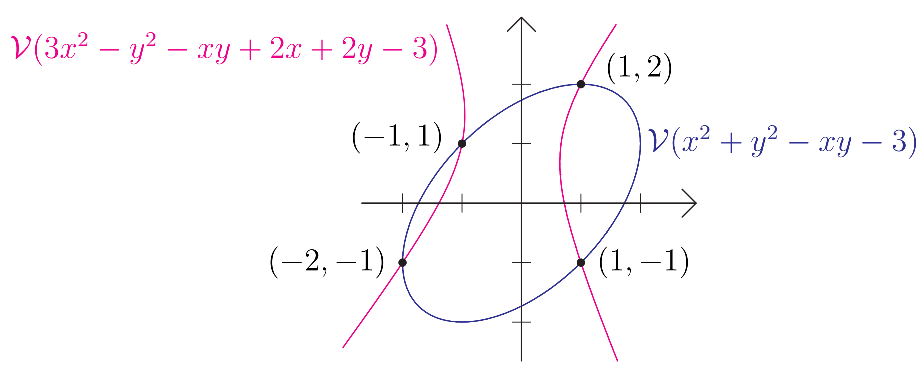


Figure 1.1: Two hyperboloids.

intersection of an ellipse $\mathcal{V}(x^2+y^2-xy-3)$ and a hyperbola $\mathcal{V}(3x^2-y^2-xy+2x+2y-3)$.



The quadrics of Figure 1.1 meet in the variety $\mathcal{V}(xy+z,\ x^2-x+y^2+yz)$, which is shown on the right in Figure 1.2. This intersection is the union of two space curves. One is the

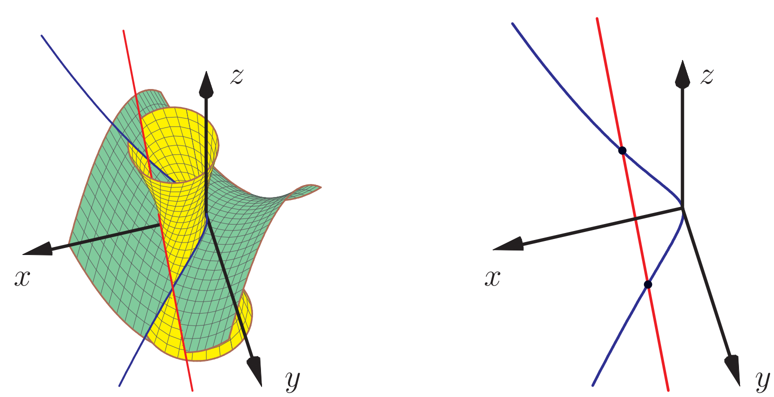


Figure 1.2: Intersection of two quadrics.

line $x = 1, y + z = 0$, while the other is the cubic space curve which has parametrization $t \mapsto (t^2, t, -t^3)$. Observe that the sum of the degrees of these curves, 1 (for the line) and 3 (for the space cubic) is equal to the product $2 \cdot 2$ of the degrees of the quadrics defining the intersection.

The intersection of the hyperboloid $x^2+(y-\frac{3}{2})^2-z^2 = \frac{1}{4}$ with the sphere $x^2+y^2+z^2 = 4$ is a singular space curve (the figure ∞ on the left sphere in Figure 1.3). If we instead intersect the hyperboloid with the sphere centered at the origin having radius 1.9, then we obtain the smooth quartic space curve drawn on the right sphere in Figure 1.3.

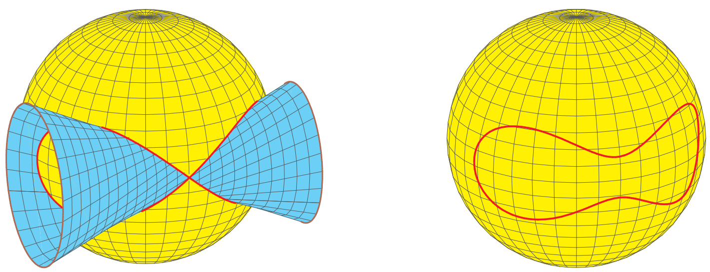


Figure 1.3: Quartics on spheres.

The product $X \times Y$ of two varieties X and Y is again a variety. Indeed, suppose that $X \subset \mathbb{K}^n$ is defined by the polynomials $f_1, \ldots, f_s \in \mathbb{K}[x_1, \ldots, x_n]$ and that $Y \subset \mathbb{K}^m$ is defined by the polynomials $g_1, \ldots, g_t \in \mathbb{K}[y_1, \ldots, y_m]$. Then $X \times Y \subset \mathbb{K}^n \times \mathbb{K}^m = \mathbb{K}^{n+m}$ is defined by the polynomials $f_1, \ldots, f_s, g_1, \ldots, g_t \in \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Given a point $x \in X$, the product $\{x\} \times Y$ is a subvariety of $X \times Y$ which may be identified with Y simply by forgetting the coordinate x.

The set $\mathrm{Mat}_{n\times n}$ or $\mathrm{Mat}_{n\times n}(\mathbb{K})$ of $n \times n$ matrices with entries in $\mathbb{K}$ is identified with the affine space $\mathbb{K}^{n^2}$, which may be written $\mathbb{K}^{n\times n}$. An interesting class of varieties are linear algebraic groups, which are algebraic subvarieties of $\mathrm{Mat}_{n\times n}$ that are closed under multiplication and taking inverses. The *special linear group* is the set of matrices with determinant 1,

$$SL_n \;:=\; \{M \in \mathrm{Mat}_{n\times n} \mid \det M = 1\}\,,$$

which is a linear algebraic group. Since the determinant of a matrix in $\mathrm{Mat}_{n\times n}$ is a polynomial in its entries, SL_n is the variety $\mathcal{V}(\det -1)$. We will later show that SL_n is smooth, irreducible, and has dimension $n^2 - 1$. (We must first, of course, define these notions.)

There is a general construction of other linear algebraic groups. Let g^T be the transpose of a matrix $g \in \mathrm{Mat}_{n\times n}$. For a fixed matrix $M \in \mathrm{Mat}_{n\times n}$, set

$$G_M \;:=\; \{g \in SL_n \mid gMg^T = M\}\,.$$

This a linear algebraic group, as the condition $gMg^T = M$ is n^2 polynomial equations in the entries of g, and G_M is closed under matrix multiplication and matrix inversion.

When M is skew-symmetric and invertible, G_M is a *symplectic group*. In this case, n is necessarily even. If we let J_n denote the $n \times n$ matrix with ones on its anti-diagonal, then the matrix

$$\begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}$$

is conjugate to every other invertible skew-symmetric matrix in $\mathrm{Mat}_{2n\times 2n}$. We assume M is this matrix and write Sp_{2n} for the symplectic group.

When M is symmetric and invertible, G_M is a *special orthogonal group*. When $\mathbb{K}$ is algebraically closed, all invertible symmetric matrices are conjugate, and we may assume $M = J_n$. For general fields, there may be many different forms of the special orthogonal group. For instance, when $\mathbb{K} = \mathbb{R}$, let k and l be, respectively, the number of positive and negative eigenvalues of M (these are conjugation invariants of M). Then we obtain the group $SO_{k,l}\mathbb{R}$. We have $SO_{k,l}\mathbb{R} \simeq SO_{l,k}\mathbb{R}$.

Consider the two extreme cases. When $l = 0$, we may take $M = I_n$, and so we obtain the special orthogonal group $SO_{n,0} = SO_n(\mathbb{R})$ of rotation matrices in $\mathbb{R}^n$, which is compact in the usual topology. The other extreme case is when $|k - l| \leq 1$, and we may take $M = J_n$. This gives the split form of the special orthogonal group which is not compact.

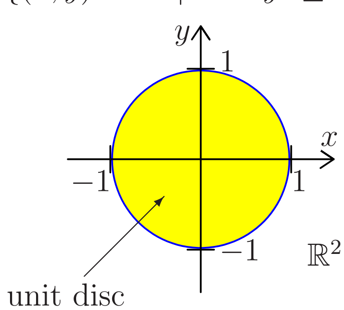
When $n = 2$, consider the two different real groups:

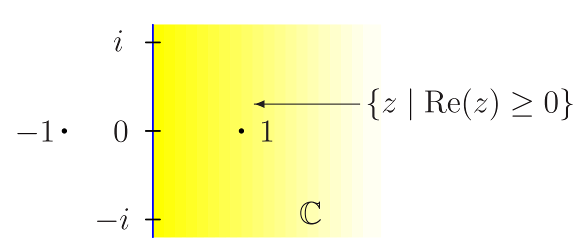
$$\begin{aligned} SO_{2,0}\mathbb{R} &:= \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \mid \theta \in S^1 \right\} \\ SO_{1,1}\mathbb{R} &:= \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mid a \in \mathbb{R}^\times \right\} \end{aligned}$$

Note that in the Euclidean topology $SO_{2,0}(\mathbb{R})$ is compact, while $SO_{1,1}(\mathbb{R})$ is not. The complex group $SO_2(\mathbb{C})$ is also not compact in the Euclidean topology.

We also point out some subsets of $\mathbb{K}^n$ which are not varieties. The set $\mathbb{Z}$ of integers is not a variety. The only polynomial vanishing at every integer is the zero polynomial, whose variety is all of $\mathbb{K}$. The same is true for any other infinite proper subset of $\mathbb{K}$, for example, the infinite sequence $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is not a subvariety of $\mathbb{K}$.

Other subsets which are not varieties (for the same reasons) include the unit disc in $\mathbb{R}^2$, $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ or the complex numbers with positive real part.





Sets like these last two which are defined by inequalities involving real polynomials are called *semi-algebraic*. We will study them in Chapter 4.

Exercises

1. Show that no proper nonempty open subset S of $\mathbb{R}^n$ or $\mathbb{C}^n$ is a variety. Here, we mean open in the usual (Euclidean) topology on $\mathbb{R}^n$ and $\mathbb{C}^n$. (Hint: Consider the Taylor expansion of any polynomial that vanishes identically on S.)

2. Suppose that $S \subset T$ are sets of polynomials in $\mathbb{K}[x_1, \ldots, x_n]$. Show that $\mathcal{V}(S) \supset \mathcal{V}(T)$.

3. Prove that in $\mathbb{K}^2$ we have $\mathcal{V}(y - x^2) = \mathcal{V}(y^3 - y^2x^2,\ x^2y - x^4)$.

4. Express the cubic space curve C with parametrization (t, t^2, t^3) in each of the following ways.

   (a) The intersection of a quadric hypersurface and a cubic hypersurface.

   (b) The intersection of two quadrics.

   (c) The intersection of three quadrics.

5. Let $\mathbb{K}^{n\times n}$ be the set of $n \times n$ matrices over $\mathbb{K}$.

   (a) Show that the set $SL(n, \mathbb{K}) \subset \mathbb{K}^{n\times n}$ of matrices with determinant 1 is an algebraic variety.

   (b) Show that the set of singular matrices in $\mathbb{K}^{n\times n}$ is an algebraic variety.

   (c) Show that the set $GL(n, \mathbb{K})$ of invertible matrices is not an algebraic variety in $\mathbb{K}^{n\times n}$. Show that $GL_n(\mathbb{K})$ can be identified with an algebraic subset of $\mathbb{K}^{n^2+1} = \mathbb{K}^{n\times n} \times \mathbb{K}^1$ via a map $GL_n(\mathbb{K}) \to \mathbb{K}^{n^2+1}$.

6. An $n\times n$ matrix with complex entries is *unitary* if its columns are orthonormal under the complex inner product $\langle z, w\rangle = z \cdot \overline{w}^t = \sum_{i=1}^n z_i \overline{w_i}$. Show that the set $\mathbf{U}(n)$ of unitary matrices is not a complex algebraic variety. Show that it can be described as the zero locus of a collection of polynomials with real coefficients in $\mathbb{R}^{2n^2}$, and so it is a real algebraic variety.

7. Let $\mathbb{K}^{m\times n}$ be the set of $m \times n$ matrices over $\mathbb{K}$.

   (a) Show that the set of matrices of rank $\leq r$ is an algebraic variety.

   (b) Show that the set of matrices of rank $= r$ is not an algebraic variety if $r > 0$.

8. (a) Show that the set $\{(t, t^2, t^3) \mid t \in \mathbb{K}\}$ is an algebraic variety in $\mathbb{K}^3$.

   (b) Show that the following sets are not algebraic varieties

      (i) $\{(x, y) \in \mathbb{R}^2 | y = \sin x\}$

      (ii) $\{(\cos t, \sin t, t) \in \mathbb{R}^3 \mid t \in \mathbb{R}\}$

      (iii) $\{(x, e^x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$

1.2 The algebraic-geometric dictionary

The strength and richness of algebraic geometry as a subject and source of tools for applications comes from its dual, simultaneously algebraic and geometric, nature. Intuitive geometric concepts are tamed via the precision of algebra while basic algebraic notions are enlivened by their geometric counterparts. The source of this dual nature is a correspondence between algebraic concepts and geometric concepts that we refer to as the algebraic-geometric dictionary.

We defined varieties $\mathcal{V}(S)$ associated to sets $S \subset \mathbb{K}[x_1, \ldots, x_n]$ of polynomials,

$$\mathcal{V}(S) \;=\; \{x \in \mathbb{K}^n \mid f(x) = 0 \text{ for all } f \in S\}\,.$$

We would like to invert this association. Given a subset Z of $\mathbb{K}^n$, consider the collection of polynomials that vanish on Z,

$$\mathcal{I}(Z) \;:=\; \{f \in \mathbb{K}[x_1, \ldots, x_n] \mid f(z) = 0 \text{ for all } z \in Z\}\,.$$

The map $\mathcal{I}$ reverses inclusions so that $Z \subset Y$ implies $\mathcal{I}(Z) \supset \mathcal{I}(Y)$.

These two inclusion-reversing maps

$$\{\text{Subsets } S \text{ of } \mathbb{K}[x_1, \ldots, x_n]\} \quad \underset{\mathcal{I}}{\overset{\mathcal{V}}{\rightleftarrows}} \quad \{\text{Subsets } Z \text{ of } \mathbb{K}^n\} \tag{1.1}$$

form the basis of the algebra-geometry dictionary of affine algebraic geometry. We will refine this correspondence to make it more precise.

An *ideal* is a subset $I \subset \mathbb{K}[x_1, \ldots, x_n]$ which is closed under addition and under multiplication by polynomials in $\mathbb{K}[x_1, \ldots, x_n]$. If $f, g \in I$ then $f + g \in I$ and if we also have $h \in \mathbb{K}[x_1, \ldots, x_n]$, then $hf \in I$. The *ideal* $\langle S \rangle$ generated by a subset S of $\mathbb{K}[x_1, \ldots, x_n]$ is the smallest ideal containing S. It is the set of all expressions of the form

$$h_1 f_1 + \cdots + h_m f_m$$

where $f_1, \ldots, f_m \in S$ and $h_1, \ldots, h_m \in \mathbb{K}[x_1, \ldots, x_n]$. We work with ideals because if f, g, and h are polynomials and $x \in \mathbb{K}^n$ with $f(x) = g(x) = 0$, then $(f + g)(x) = 0$ and $(hf)(x) = 0$. Thus $\mathcal{V}(S) = \mathcal{V}(\langle S \rangle)$, and so we may restrict $\mathcal{V}$ to the ideals of $\mathbb{K}[x_1, \ldots, x_n]$. In fact, we lose nothing if we restrict the left-hand-side of the correspondence (1.1) to the ideals of $\mathbb{K}[x_1, \ldots, x_n]$.

Lemma 1.2.1. *For any subset S of $\mathbb{K}^n$, $\mathcal{I}(S)$ is an ideal of $\mathbb{K}[x_1, \ldots, x_n]$.*

Proof. Let $f, g \in \mathcal{I}(S)$ be two polynomials which vanish at all points of S. Then $f + g$ vanishes on S, as does hf, where h is any polynomial in $\mathbb{K}[x_1, \ldots, x_n]$. This shows that $\mathcal{I}(S)$ is an ideal of $\mathbb{K}[x_1, \ldots, x_n]$. $\square$

When S is infinite, the variety $\mathcal{V}(S)$ is defined by infinitely many polynomials. Hilbert's Basis Theorem tells us that only finitely many of these polynomials are needed.

Hilbert's Basis Theorem. *Every ideal I of $\mathbb{K}[x_1,\ldots,x_n]$ is finitely generated.*

We will prove this in Chapter 2. Be more specific!

Hilbert's Basis Theorem implies many important finiteness properties of algebraic varieties.

Corollary 1.2.2. *Any variety $Z\subset\mathbb{K}^n$ is the intersection of finitely many hypersurfaces.*

Proof. Let $Z=\mathcal{V}(I)$ be defined by the ideal I. By Hilbert's Basis Theorem, I is finitely generated, say by $f_1,\ldots,f_s$, and so $Z=\mathcal{V}(f_1,\ldots,f_s)=\mathcal{V}(f_1)\cap\cdots\cap\mathcal{V}(f_s)$. $\square$

Example 1.2.3. The ideal of the cubic space curve C of Figure 1.2 with parametrization $(t^2,-t,t^3)$ not only contains the polynomials $xy+z$ and x^2-x+y^2+yz, but also y^2-x, x^2+yz, and y^3+z. Not all of these polynomials are needed to define C as $x^2-x+y^2+yz=(y^2-x)+(x^2+yz)$ and $y^3+z=y(y^2-x)+(xy+z)$. In fact three of the quadrics suffice,

$$\mathcal{I}(C) \;=\; \langle xy+z,\ y^2-x,\ x^2+yz\rangle\,.$$

Lemma 1.2.4. *For any subset Z of $\mathbb{K}^n$, if $X=\mathcal{V}(\mathcal{I}(Z))$ is the variety defined by the ideal $\mathcal{I}(Z)$, then $\mathcal{I}(X)=\mathcal{I}(Z)$ and X is the smallest variety containing Z.*

Proof. Set $X:=\mathcal{V}(\mathcal{I}(Z))$. Then $\mathcal{I}(Z)\subset\mathcal{I}(X)$, since if f vanishes on Z, it will vanish on X. However, $Z\subset X$, and so $\mathcal{I}(Z)\supset\mathcal{I}(X)$, and thus $\mathcal{I}(Z)=\mathcal{I}(X)$.

If Y was a variety with $Z\subset Y\subset X$, then $\mathcal{I}(X)\subset\mathcal{I}(Y)\subset\mathcal{I}(Z)=\mathcal{I}(X)$, and so $\mathcal{I}(Y)=\mathcal{I}(X)$. But then we must have $Y=X$ for otherwise $\mathcal{I}(X)\subsetneq\mathcal{I}(Y)$, as is shown in Exercise 3. $\square$

Thus we also lose nothing if we restrict the right-hand-side of the correspondence (1.1) to the subvarieties of $\mathbb{K}^n$. Our correspondence now becomes

$$\{\text{Ideals } I \text{ of } \mathbb{K}[x_1,\ldots,x_n]\} \quad \underset{\mathcal{I}}{\overset{\mathcal{V}}{\rightleftarrows}} \quad \{\text{Subvarieties } X \text{ of } \mathbb{K}^n\}\ . \tag{1.2}$$

This association is not a bijection. In particular, the map $\mathcal{V}$ is not one-to-one and the map $\mathcal{I}$ is not onto. There are several reasons for this.

For example, when $\mathbb{K}=\mathbb{Q}$ and $n=1$, we have $\emptyset=\mathcal{V}(1)=\mathcal{V}(x^2-2)$. The problem here is that the rational numbers are not algebraically closed and we need to work with a larger field (for example $\mathbb{Q}(\sqrt{2})$) to study $\mathcal{V}(x^2-2)$. When $\mathbb{K}=\mathbb{R}$ and $n=1$, $\emptyset\neq\mathcal{V}(x^2-2)$, but we have $\emptyset=\mathcal{V}(1)=\mathcal{V}(1+x^2)=\mathcal{V}(1+x^4)$. While the problem here is again that the real numbers are not algebraically closed, we view this as a manifestation of positivity. The two polynomials $1+x^2$ and $1+x^4$ only take positive values. When working over $\mathbb{R}$ (as our interest in applications leads us to do so) positivity of polynomials plays an important role, as we will see in later chapters.

The problem with the map $\mathcal{V}$ is more fundamental than these examples reveal and occurs even when $\mathbb{K} = \mathbb{C}$. When $n = 1$ we have $\{0\} = \mathcal{V}(x) = \mathcal{V}(x^2)$, and when $n = 2$, we invite the reader to check that $\mathcal{V}(y - x^2) = \mathcal{V}(y^2 - yx^2, xy - x^3)$. Note that while $x \notin \langle x^2\rangle$, we have $x^2 \in \langle x^2\rangle$. Similarly, $y - x^2 \notin \mathcal{V}(y^2 - yx^2, xy - x^3)$, but

$$(y - x^2)^2 \;=\; y^2 - yx^2 \;-\; x(xy - x^3) \;\in\; \langle y^2 - yx^2, xy - x^3\rangle\,. \tag{1.3}$$

In both cases, the lack of injectivity of the map $\mathcal{V}$ is because f and f^N have the same set of zeroes, for any positive integer N. For example, if $f_1, \dots, f_s$ are polynomials, then the two ideals

$$\langle f_1,\, f_2,\, \dots,\, f_s\rangle \qquad \text{and} \qquad \langle f_1,\, f_2^2,\, f_3^3,\, \dots,\, f_s^s\rangle$$

both define the same variety, and if $f^N \in \mathcal{I}(Z)$, then $f \in \mathcal{I}(Z)$.

We clarify this point with a definition. An ideal $I \subset \mathbb{K}[x_1, \dots, x_n]$ is *radical* if whenever $f^N \in I$ for some positive integer N, then $f \in I$. The radical $\sqrt{I}$ of an ideal I of $\mathbb{K}[x_1, \dots, x_n]$ is

$$\sqrt{I} \;:=\; \{f \in \mathbb{K}[x_1, \dots, x_n] \mid f^N \in I\,,\ \text{for some } N \geq 1\}\,.$$

You will show in Exercise 2 that $\sqrt{I}$ is the smallest radical ideal containing I. For example (1.3) shows that

$$\sqrt{\langle y^2 - yx^2, xy - x^3\rangle} \;=\; \langle y - x^2\rangle\,.$$

The reason for this definition is twofold: first, $\mathcal{I}(Z)$ is radical, and second, an ideal I and its radical $\sqrt{I}$ both define the same variety. We record these facts.

Lemma 1.2.5. *For $Z \subset \mathbb{K}^n$, $\mathcal{I}(Z)$ is a radical ideal. If $I \subset \mathbb{K}[x_1, \dots, x_n]$ is an ideal, then $\mathcal{V}(I) = \mathcal{V}(\sqrt{I})$.*

When $\mathbb{K}$ is algebraically closed, the precise nature of the correspondence (1.2) follows from Hilbert's Nullstellensatz (null=zeroes, stelle=places, satz=theorem), another of Hilbert's foundational results in the 1890's that helped to lay the foundations of algebraic geometry and usher in twentieth century mathematics. We first state a weak form of the Nullstellensatz, which describes the ideals defining the empty set.

Theorem 1.2.6 (Weak Nullstellensatz). *Suppose that $\mathbb{K}$ is algebraically closed. If I is an ideal of $\mathbb{K}[x_1, \dots, x_n]$ with $\mathcal{V}(I) = \emptyset$, then $I = \mathbb{K}[x_1, \dots, x_n]$.*

Let $b = (b_1, \dots, b_n) \in \mathbb{K}^n$. Then $\{b\}$ is defined by the linear polynomials $x_i - b_i$ for $i = 1, \dots, n$. A polynomial f is equal to the constant $f(b)$ modulo the ideal $\mathfrak{m}_b := \langle x_1 - b_1, \dots, x_n - b_n\rangle$ generated by these polynomials, thus the quotient ring $\mathbb{K}[x_1, \dots, x_n]/\mathfrak{m}_b$ is isomorphic to the field $\mathbb{K}$ and so $\mathfrak{m}_b$ is a maximal ideal. In fact, these are the only maximal ideals.

Theorem 1.2.7. *Every maximal* $\mathfrak{m}$ *ideal of* $\mathbb{K}[x_1,\ldots,x_n]$ *has the form* $\mathfrak{m}_b$ *for some* $b \in \mathbb{K}^n$.

Proof. We prove this when $\mathbb{K}$ is uncountable field, e.g. $\mathbb{K} = \mathbb{C}$. Then $\mathbb{K}[x_1,\ldots,x_n]/\mathfrak{m}$ is a field, L that contains $\mathbb{K}$ whose dimension as a $\mathbb{K}$-vector space is at most countable (it is spanned by the images of the monomials). Since $\mathbb{K}$ is algebraically closed, we have $L \neq \mathbb{K}$ only if L contains an element that is transcendental over $\mathbb{K}$. But then L contains a subfield isomorphic to the field $\mathbb{K}(t)$ of rational functions in t. Consider the uncountable subset of $\mathbb{K}(t)$,

$$\left\{\frac{1}{t-a} \mid a \in \mathbb{K}\right\}.$$

We claim that this set is linearly independent. If we had a linear dependency,

$$0 \;=\; \sum_{i=1}^{m} \lambda_i \frac{1}{t-a_i}\,,$$

then we could multiply it by $(t-a_i)$, simplify, and substitute $t = a_i$ to find that $\lambda_i = 0$, for every i. Thus $\mathbb{K}(t)$ has uncountable dimension over $\mathbb{K}$ and so L cannot contain a subfield isomorphic to $\mathbb{K}(t)$.

Thus we conclude that $L = \mathbb{K}$. If $b_i \in \mathbb{K}$ is the image of the variable x_i, then we see that $\mathfrak{m} \supset \mathfrak{m}_b$. As these are maximal ideals, they are in fact equal. □

Proof of the weak Nullstellensatz. We prove the contrapositive, if $I \subsetneq \mathbb{C}[x_1,\ldots,x_n]$ is a proper ideal, then $\mathcal{V}(I) \neq \emptyset$. There is a maximal ideal $\mathfrak{m}_b$ with $b \in \mathbb{K}^n$ of $\mathbb{C}[x_1,\ldots,x_n]$ which contains I. But then

$$\{b\} \;=\; \mathcal{V}(\mathfrak{m}_b) \;\subset\; \mathcal{V}(I)\,,$$

and so $\mathcal{V}(I) \neq \emptyset$. Thus if $\mathcal{V}(I) = \emptyset$, we must have $I = \mathbb{C}[x_1,\ldots,x_n]$, which proves the weak Nullstellensatz. □

The Fundamental Theorem of Algebra states that any nonconstant polynomial $f \in \mathbb{C}[x]$ has a root (a solution to $f(x) = 0$). We recast the weak Nullstellensatz as the multivariate fundamental theorem of algebra.

Theorem 1.2.8 (Multivariate Fundamental Theorem of Algebra)**.** *If the polynomials* $f_1,\ldots,f_m \in \mathbb{C}[x_1,\ldots,x_n]$ *generate a proper ideal of* $\mathbb{C}[x_1,\ldots,x_n]$, *then the system of polynomial equations*

$$f_1(x) \;=\; f_2(x) \;=\; \cdots \;=\; f_m(x) \;=\; 0$$

has a solution in $\mathbb{K}^n$.

We now deduce the strong Nullstellensatz, which we will use to complete the characterization (1.2).

Theorem 1.2.9 (Nullstellensatz). *If $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is an ideal, then $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$.*

Proof. Since $\mathcal{V}(I) = \mathcal{V}(\sqrt{I})$, we have $\sqrt{I} \subset \mathcal{I}(\mathcal{V}(I))$. We show the other inclusion. Suppose that we have a polynomial $f \in \mathcal{I}(\mathcal{V}(I))$. Introduce a new variable t. Then the variety $\mathcal{V}(I, tf-1) \subset \mathbb{K}^{n+1}$ defined by I and $tf-1$ is empty. Indeed, if $(a_1, \ldots, a_n, b)$ were a point of this variety, then $(a_1, \ldots, a_n)$ would be a point of $\mathcal{V}(I)$. But then $f(a_1, \ldots, a_n) = 0$, and so the polynomial $tf-1$ evaluates to 1 (and not 0) at the point $(a_1, \ldots, a_n, b)$.

By the weak Nullstellensatz, $\langle I, tf-1\rangle = \mathbb{C}[x_1, \ldots, x_n, t]$. In particular, $1 \in \langle I, tf-1\rangle$, and so there exist polynomials $f_1, \ldots, f_m \in I$ and $g, g_1, \ldots, g_m \in \mathbb{C}[x_1, \ldots, x_n, t]$ such that

$$1 \;=\; f_1(x)g_1(x,t) + f_2(x)g_2(x,t) + \cdots + f_m(x)g_m(x,t) \;+\; g(x,t)(tf(x)-1)\,.$$

If we apply the substitution $t = \frac{1}{f}$, then the last term with factor $tf-1$ vanishes and each polynomial $g_i(x,t)$ becomes a rational function in $x_1, \ldots, x_n$ whose denominator is a power of f. Clearing these denominators gives an expression of the form

$$f^N \;=\; f_1(x)G_1(x) + f_2(x)G_2(x) + \cdots + f_m(x)G_m(x)\,,$$

where $G_1, \ldots, G_m \in \mathbb{C}[x_1, \ldots, x_n]$. But this shows that $f \in \sqrt{I}$, and completes the proof of the Nullstellensatz. $\square$

Corollary 1.2.10 (Algebraic-Geometric Dictionary I). *Over any field $\mathbb{K}$, the maps $\mathcal{V}$ and $\mathcal{I}$ give an inclusion reversing correspondence*

$$\{\textit{Radical ideals } I \textit{ of } \mathbb{K}[x_1, \ldots, x_n]\} \quad \underset{\mathcal{I}}{\overset{\mathcal{V}}{\rightleftarrows}} \quad \{\textit{Subvarieties } X \textit{ of } \mathbb{K}^n\} \tag{1.4}$$

with $\mathcal{V}(\mathcal{I}(X)) = X$. When $\mathbb{K}$ is algebraically closed, the maps $\mathcal{V}$ and $\mathcal{I}$ are inverses, and this correspondence is a bijection.

Proof. First, we already observed that $\mathcal{I}$ and $\mathcal{V}$ are reverse inclusions and these maps have the domain and range indicated. Let X be a subvariety of $\mathbb{K}^n$. In Lemma 1.2.4 we showed that $X = \mathcal{V}(\mathcal{I}(X))$. Thus $\mathcal{V}$ is onto and $\mathcal{I}$ is one-to-one.

Now suppose that $\mathbb{K}$ is algebraically closed. By the Nullstellensatz, if I is radical then $\mathcal{I}(\mathcal{V}(I)) = I$, and so $\mathcal{I}$ is onto and $\mathcal{V}$ is one-to-one. This shows that $\mathcal{I}$ and $\mathcal{V}$ are inverse bijections. $\square$

Corollary 1.2.10 is only the beginning of the algebraic-geometric dictionary. Many natural operations on varieties correspond to natural operations on their ideals. The *sum* $I + J$ and *product* $I \cdot J$ of ideals I and J are defined to be

$$\begin{aligned} I + J &:= \{f + g \mid f \in I \quad \text{and} \quad g \in J\} \\ I \cdot J &:= \langle f \cdot g \mid f \in I \quad \text{and} \quad g \in J\rangle\,. \end{aligned}$$

Lemma 1.2.11. *Let I, J be ideals in $\mathbb{K}[x_1, \ldots, x_n]$ and set $X := \mathcal{V}(I)$ and $Y := \mathcal{V}(J)$ to be their corresponding varieties. Then*

1. $\mathcal{V}(I+J) = X \cap Y$,
2. $\mathcal{V}(I \cdot J) = \mathcal{V}(I \cap J) = X \cup Y$,

If $\mathbb{K}$ is algebraically closed, then we also have

3. $\mathcal{I}(X \cap Y) = \sqrt{I+J}$, *and*
4. $\mathcal{I}(X \cup Y) = \sqrt{I \cap J} = \sqrt{I \cdot J}$.

Example 1.2.12. It can happen that $I \cdot J \neq I \cap J$. For example, if $I = \langle xy - x^3 \rangle$ and $J = \langle y^2 - x^2y \rangle$, then $I \cdot J = \langle xy(y-x^2)^2 \rangle$, while $I \cap J = \langle xy(y-x^2) \rangle$.

This correspondence will be further refined in Section 1.3 to include maps between varieties. Because of this correspondence, each geometric concept has a corresponding algebraic concept, and *vice-versa*, when $\mathbb{K}$ is algebraically closed. When $\mathbb{K}$ is not algebraically closed, this correspondence is not exact. In that case we will often use algebra to guide our geometric definitions.

Exercises

1. Verify the claim in the text that the smallest ideal containing a set $S \subset \mathbb{K}[x_1, \ldots, x_n]$ of polynomials consists of all expressions of the form
$$h_1 f_1 + \cdots + h_m f_m$$
where $f_1, \ldots, f_m \in S$ and $h_1, \ldots, h_m \in \mathbb{K}[x_1, \ldots, x_n]$.

2. Let I be an ideal of $\mathbb{K}[x_1, \ldots, x_n]$. Show that
$$\sqrt{I} \ := \ \{f \in \mathbb{K}[x_1, \ldots, x_n] \mid f^N \in I, \text{ for some } N \in \mathbb{N}\}$$
is an ideal, is radical, and is the smallest radical ideal containing I.

3. If $Y \subsetneq X$ are varieties, show that $\mathcal{I}(X) \subsetneq \mathcal{I}(Y)$.

4. Suppose that I and J are radical ideals. Show that $I \cap J$ is also a radical ideal.

5. Give radical ideals I and J for which $I + J$ is not radical.

6. Let I be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$. Prove or find counterexamples to the following statements. Make your assumptions clear.
   (a) If $\mathcal{V}(I) = \mathbb{K}^n$ then $I = \langle 0 \rangle$.
   (b) If $\mathcal{V}(I) = \emptyset$ then $I = \mathbb{K}[x_1, \ldots, x_n]$.

7. Give two algebraic varieties Y and Z such that $\mathcal{I}(Y \cap Z) \neq \mathcal{I}(Y) + \mathcal{I}(Z)$.

8. (a) Let I be an ideal of $\mathbb{K}[x_1, \ldots, x_n]$. Show that if $\mathbb{K}[x_1, \ldots, x_n]/I$ is a finite dimensional $\mathbb{K}$-vector space then $\mathcal{V}(I)$ is a finite set.

   (b) Let $J = \langle xy, yz, xz \rangle$ be an ideal in $\mathbb{K}[x, y, z]$. Find the generators of $\mathcal{I}(\mathcal{V}(J))$. Show that J cannot be generated by two polynomials in $\mathbb{K}[x, y, z]$. Describe $V(I)$ where $I = \langle xy, xz - yz \rangle$. Show that $\sqrt{I} = J$.

9. Let $f, g \in \mathbb{K}[x, y]$ be polynomials without a common factor. Use Exercise 8(a) to show that $\mathcal{V}(f) \cap \mathcal{V}(g)$ is a finite set.

10. Prove that there are three points p, q, and r in $\mathbb{K}^2$ such that

$$\sqrt{\langle x^2 - 2xy^4 + y^6, y^3 - y \rangle} \;=\; I(\{p\}) \cap I(\{q\}) \cap I(\{r\})\,.$$

Show directly that the ideal $\langle x^2 - 2xy^4 + y^6, y^3 - y \rangle$ is not radical.

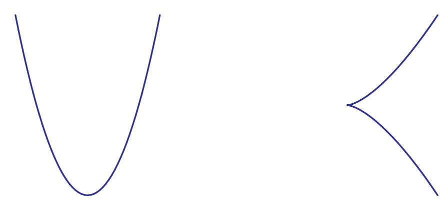
1.3 The algebrac-geometric dictionary II

We strengthen the algebra-geometry dictionary of Section 1.2 in two ways. We first replace affine space $\mathbb{K}^n$ by an affine variety X and the polynomial ring by the ring $\mathbb{K}[X]$ of regular functions on X and establish a correspondence between subvarieties of X and radical ideals of $\mathbb{K}[X]$. Next, we establish a correspondence between regular maps of varities and homomorphisms of their coordinate rings.

Let $X \subset \mathbb{K}^n$ be an affine variety and suppose that $\mathbb{K}$ is infinite. Any polynomial function $f \in \mathbb{K}[x_1, \ldots, x_n]$ restricts to give a *regular function* on X, $f\colon X \to \mathbb{K}$. We may add and multiply regular functions, and the set of all regular functions on X forms a ring, $\mathbb{K}[X]$, called the *coordinate ring* of the affine variety X or the ring of regular functions on X. The coordinate ring of an affine variety X is a basic invariant of X, which we will show is in fact equivalent to X itself.

The restriction of polynomial functions on $\mathbb{K}^n$ to regular functions on X defines a surjective ring homomorphism $\mathbb{K}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{K}[X]$. The kernel of this restriction homomorphism is the set of polynomials that vanish identically on X, that is, the ideal $\mathcal{I}(X)$ of X. Under the correspondence between ideals, quotient rings, and homomorphisms, this restriction map gives an isomorphism between $\mathbb{K}[X]$ and the quotient ring $\mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}(X)$.

Example 1.3.1. The coordinate ring of the parabola $y = x^2$ is $\mathbb{K}[x, y]/\langle y - x^2\rangle$, which is isomorphic to $\mathbb{K}[x]$, the coordinate ring of $\mathbb{K}^1$. To see this, observe that substituting x^2 for y rewrites and polynomial $f(x, y)$ as a polynomial $g(x)$ in x alone, and $y - x^2$ divides the difference $f(x, y) - g(x)$.



On the other hand, the coordinate ring of the cuspidal cubic $y^2 = x^3$ is $\mathbb{K}[x, y]/\langle y^2 - x^3\rangle$. This ring is not isomorphic to $\mathbb{K}[x, y]/\langle y - x^2\rangle$. Indeed, the element $y^2 = x^3$ has two distinct factorizations into indecomposable elements, while polynomials $f(x)$ in one variable always factor uniquely.

Let X be a variety. Its coordinate ring $\mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}(X)$ is finitely generated by the images of the variables x_i. Since $\mathcal{I}(X)$ is radical, Exercise 4 implies that this quotient ring has no nilpotent elements (elements f such that $f^M = 0$ for some M). Such a ring with no nilpotents is called *reduced*. When $\mathbb{K}$ is algebraically closed, these two properties characterize coordinate rings of algebraic varieties.

Theorem 1.3.2. *Suppose that $\mathbb{K}$ is algebraically closed. Then a $\mathbb{K}$-algebra R is the coordinate ring of an affine variety if and only if R is finitely generated and reduced.*

Proof. We need only show that a finitely generated reduced $\mathbb{K}$-algebra R is the coordinate ring of some affine variety. Suppose that the reduced $\mathbb{K}$-algebra R has generators $r_1, \ldots, r_n$. Then there is a surjective ring homomorphism

$$\varphi : \mathbb{K}[x_1, \ldots, x_n] \twoheadrightarrow R$$

given by $x_i \mapsto r_i$. Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be the kernel of φ. This identifies R with $\mathbb{K}[x_1, \ldots, x_n]/I$. Since R is reduced, we see that I is radical.

As $\mathbb{K}$ is algebraically closed, the algebraic-geometric dictionary of Corollary 1.2.10 shows that $I = \mathcal{I}(\mathcal{V}(I))$ and so $R \simeq \mathbb{K}[x_1, \ldots, x_n]/I \simeq \mathbb{K}[\mathcal{V}(I)]$. □

A different choice $s_1, \ldots, s_m$ of generators for R in this proof will give a different affine variety with the same coordinate ring R. One goal of this section is to understand this apparent ambiguity.

Example 1.3.3. The finitely generated $\mathbb{K}$-algebra $R := \mathbb{K}[t]$ is the coordinate ring of the affine line $\mathbb{K}$. Note that if we set $x := t + 1$ and $y := t^2 + 3t$, these generate R. As $y = x^2 + x - 2$, this choice of generators realizes R as $\mathbb{K}[x, y]/\langle y - x^2 - x + 2\rangle$, which is the coordinate ring of a parabola.

Among the coordinate rings $\mathbb{K}[X]$ of affine varieties are the polynomial algebras $\mathbb{K}[x_1, \ldots, x_n]$. Many properties of polynomial algebras, including the algebraic-geometric dictionary of Corollary 1.2.10 and the Hilbert Theorems hold for these coordinate rings $\mathbb{K}[X]$.

Given regular functions $f_1, \ldots, f_m \in \mathbb{K}[X]$ on an affine variety $X \subset \mathbb{K}^n$, their set of common zeroes

$$\mathcal{V}(f_1, \ldots, f_m) := \{x \in X \mid f_1(x) = \cdots = f_m(x) = 0\},$$

is a subvariety of X. To see this, let $F_1, \ldots, F_m \in \mathbb{K}[x_1, \ldots, x_n]$ be polynomials which restrict to the functions $f_1, \ldots, f_m$ on X. Then

$$\mathcal{V}(f_1, \ldots, f_m) = X \cap \mathcal{V}(F_1, \ldots, F_m),$$

and we recall that intersecrtions of varieties are again varieties. As in Section 1.2, we may extend this notation and define $\mathcal{V}(I)$ for an ideal I of $\mathbb{K}[X]$. If $Y \subset X$ is a subvariety of X, then $\mathcal{I}(X) \subset \mathcal{I}(Y)$ and so $\mathcal{I}(Y)/\mathcal{I}(X)$ is an ideal in the coordinate ring $\mathbb{K}[X] = \mathbb{K}[\mathbb{K}^n]/\mathcal{I}(X)$ of X. Write $\mathcal{I}(Y) \subset \mathbb{K}[X]$ for the ideal of Y in $\mathbb{K}[X]$.

Both Hilbert's Basis Theorem and Hilbert's Nullstellensätze have analogs for affine varieties X and their coordinate rings $\mathbb{K}[X]$. These consequences of the original Hilbert Theorems follow from the surjection $\mathbb{K}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{K}[X]$ and corresponding inclusion $X \hookrightarrow \mathbb{K}^n$.

Theorem 1.3.4 (Hilbert Theorems for $\mathbb{K}[X]$)**.** *Let X be an affine variety. Then*

1. *Any ideal of $\mathbb{K}[X]$ is finitely generated.*

2. *If Y is a subvariety of X then $\mathcal{I}(Y) \subset \mathbb{K}[X]$ is a radical ideal.*
3. *Suppose that $\mathbb{K}$ is algebraically closed. An ideal I of $\mathbb{K}[X]$ defines the empty set if and only if $I = \mathbb{K}[X]$.*

As in Section 1.2 we obtain a version of the algebraic-geometric dictionary between subvarieties of an affine variety X and radical ideals of $\mathbb{K}[X]$. The proofs are nearly the same, so we leave them to the reader. For this, you will need to recall that ideals of a quotient ring R/I all have the form J/I, where J is an ideal of R which contains I.

Theorem 1.3.5. *Let X be an affine variety. Then the maps $\mathcal{V}$ and $\mathcal{I}$ give an inclusion reversing correspondence*

$$\{\textit{Radical ideals } I \textit{ of } \mathbb{K}[X]\} \quad \underset{\mathcal{I}}{\overset{\mathcal{V}}{\rightleftarrows}} \quad \{\textit{Subvarieties } Y \textit{ of } X\} \tag{1.5}$$

with $\mathcal{I}$ injective and $\mathcal{V}$ surjective. When $\mathbb{K}$ is algebraically closed, the maps $\mathcal{V}$ and $\mathcal{I}$ are inverse bijections.

We do not just study varieties, but also the maps between them.

Definition 1.3.6. A list $f_1, \ldots, f_m \in \mathbb{K}[X]$ of regular functions on an affine variety X defines a function

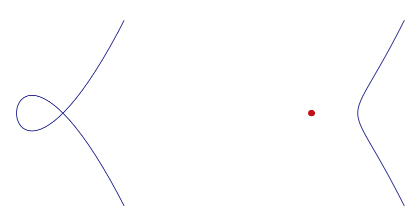
$$\begin{array}{rcl} \varphi : X & \longrightarrow & \mathbb{K}^m \\ x & \longmapsto & (f_1(x), f_2(x), \ldots, f_m(x))\,, \end{array}$$

which we call a *regular map*.

Example 1.3.7. The elements $t^2, t, -t^3 \in \mathbb{K}[t]$ define the map $\mathbb{K}^1 \to \mathbb{K}^3$ whose image is the cubic curve of Figure 1.2.

The elements t^2, t^3 of $\mathbb{K}[t]$ define a map $\mathbb{K}^1 \to \mathbb{K}^2$ whose image is the cuspidal cubic that we saw earlier.

Let $x = t^2 - 1$ and $y = t^3 - t$, which are elements of $\mathbb{K}[t]$. These define a map $\mathbb{K}^1 \to \mathbb{K}^2$ whose image is the nodal cubic curve $\mathcal{V}(y^2 - (x^3 + x^2))$ on the left below. If we instead take $x = t^2 + 1$ and $y = t^3 + t$, then we get a different map $\mathbb{K}^1 \to \mathbb{K}^2$ whose image is the curve $\mathcal{V}(y^2 - (x^3 - x^2))$ on the right below.



In the curve on the right, the image of $\mathbb{R}^1$ is the arc, while the isolated or *solitary point* is the image of the points $\pm\sqrt{-1}$.

Suppose that X is an affine variety and we have a regular map $\varphi\colon X \to \mathbb{K}^m$ given by regular functions $f_1, \dots, f_m \in \mathbb{K}[X]$. A polynomial $g \in \mathbb{K}[x_1, \dots, x_m]$ *pulls back* along φ to give the regular function $\varphi^* g$, which is defined by

$$\varphi^* g \;:=\; g(f_1, \dots, f_m)\,.$$

This element of the coordinate ring $\mathbb{K}[X]$ of X is the usual pull back of a function. For $x \in X$ we have

$$(\varphi^* g)(x) \;=\; g(\varphi(x)) \;=\; g(f_1(x), \dots, f_m(x))\,.$$

The resulting map $\varphi^*\colon \mathbb{K}[x_1, \dots, x_m] \to \mathbb{K}[X]$ is a homomorphism of $\mathbb{K}$-algebras. Conversely, given a homomorphism $\psi\colon \mathbb{K}[x_1, \dots, x_m] \to \mathbb{K}[X]$ of $\mathbb{K}$-algebras, if we set $f_i := \psi(x_i)$, then $f_1, \dots, f_m \in \mathbb{K}[X]$ define a regular map φ with $\varphi^* = \psi$.

We have just shown the following basic fact.

Lemma 1.3.8. *The association $\varphi \mapsto \varphi^*$ defines a bijection*

$$\left\{ \begin{array}{c} \textit{Regular maps} \\ \varphi\colon X \to \mathbb{K}^m \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \mathbb{K}\textit{-algebra homomorphisms} \\ \psi\colon \mathbb{K}[x_1, \dots, x_m] \to \mathbb{K}[X] \end{array} \right\}$$

In the examples that we gave, the image $\varphi(X)$ of X under φ was contained in a subvariety. This is always the case.

Lemma 1.3.9. *Let X be an affine variety, $\varphi\colon X \to \mathbb{K}^m$ a regular map, and $Y \subset \mathbb{K}^m$ a subvariety. Then $\varphi(X) \subset Y$ if and only if $\mathcal{I}(Y) \subset \ker \varphi^*$.*

In particular, $\mathcal{V}(\ker \varphi^*)$ is the smallest subvariety of $\mathbb{K}^m$ that contains the image $\varphi(X)$ of X under φ.

Proof. First suppose that $\varphi(X) \subset Y$. If $f \in \mathcal{I}(Y)$ then f vanishes on Y and hence on $\varphi(X)$. But then $\varphi^* f$ is the zero function, and so $\mathcal{I}(Y) \subset \ker \varphi^*$.

For the other direction, suppose that $\mathcal{I}(Y) \subset \ker \varphi^*$ and let $x \in X$. If $f \in \mathcal{I}(Y)$, then $\varphi^* f = 0$ and so $0 = \varphi^* f(x) = f(\varphi(x))$. This implies that $\varphi(x) \in Y$, and so we conclude that $\varphi(X) \subset Y$. $\square$

Definition 1.3.10. Affine varieties X and Y are *isomorphic* if there are regular maps $\varphi\colon X \to Y$ and $\psi\colon Y \to X$ such that both $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity maps on Y and X, repsectively. In this case, we say that φ and ψ are isomorphisms.

Corollary 1.3.11. *Let X be an affine variety, $\varphi\colon X \to \mathbb{K}^m$ a regular map, and $Y \subset \mathbb{K}^m$ a subvariety. Then*

(1) $\ker \varphi^*$ *is a radical ideal.*

(2) $\mathcal{V}(\ker \varphi^*)$ *is the smallest affine variety containing $\varphi(X)$.*

(3) *If* $\varphi\colon X \to Y$*, then* $\varphi^*\colon \mathbb{K}[\mathbb{K}^m] \to \mathbb{K}[X]$ *factors through* $\mathbb{K}[Y]$ *inducing a homomorphism* $\mathbb{K}[Y] \to \mathbb{K}[X]$.

(4) φ *is an isomorphism of varieties if and only if* φ^* *is an isomorphism of* $\mathbb{K}$*-algebras.*

We write φ^* for the induced map $\mathbb{K}[Y] \to \mathbb{K}[X]$ of part (4).

Proof. For (1), suppose that $f^N \in \ker\varphi^*$, so that $0 = \varphi^*(f^N) = (\varphi^*(f))^N$. Since $\mathbb{K}[X]$ has no nilpotent elements, we conclude that $\varphi^*(f) = 0$ and so $f \in \ker\varphi^*$.

Suppose that Y is an affine variety containing $\varphi(X)$. By Lemma 1.3.9, $\mathcal{I}(Y) \subset \ker\varphi^*$ and so $\mathcal{V}(\ker\varphi^*) \subset Y$. Statement (2) follows as we also have $X \subset \mathcal{V}(\ker\varphi^*)$.

For (3), we have $\mathcal{I}(Y) \subset \ker\varphi^*$ and so the map $\varphi^*\colon \mathbb{K}[x_1,\ldots,x_m] \to \mathbb{K}[X]$ factors through the quotient map $\mathbb{K}[x_1,\ldots,x_m] \twoheadrightarrow \mathbb{K}[x_1,\ldots,x_m]/\mathcal{I}(Y) = \mathbb{K}[Y]$.

Statement (4) is immediate from the definitions. □

Thus we may refine the correspondence of Lemma 1.3.8. Let X and Y be affine varieties. Then the association $\varphi \mapsto \varphi^*$ gives a bijective correspondence

$$\left\{\begin{array}{c}\text{Regular}\\ \text{maps}\\ \varphi\colon X \to Y\end{array}\right\} \quad\longleftrightarrow\quad \left\{\begin{array}{c}\mathbb{K}\text{-algebra homomorphisms}\\ \psi\colon \mathbb{K}[Y] \to \mathbb{K}[X]\end{array}\right\}.$$

This map $X \mapsto \mathbb{K}[X]$ from affine varieties to finitely generated reduced $\mathbb{K}$-algebras not only sends objects to objects, but it induces an isomorphism on maps between objects (reversing their direction however). In mathematics, such an association is called a *contravariant equivalence of categories*. The point of this equivalence is that an affine variety and its coordinate ring are different packages for the same information. Each one determines and is determined by the other. Whether we study algebra or geometry, we are studying the same thing.

The prototypical example of a contravariant equivalence of categories comes from linear algebra. To a finite-dimensional vector space V, we may associate its dual space V^*. Given a linear transformation $L\colon V \to W$, its adjoint is a map $L^*\colon W^* \to V^*$. Since $(V^*)^* = V$ and $(L^*)^* = L$, this association is a bijection on the objects (finite-dimensional vector spaces) and a bijection on linear maps linear maps from V to W.

Exercises

1. Give a proof of Theorem 1.3.4.

2. Let $V = \mathcal{V}(y - x^2) \subset \mathbb{K}^2$ and $W = \mathcal{V}(xy - 1) \subset \mathbb{K}^2$. Show that

$$\begin{aligned}\mathbb{K}[V] &:= \mathbb{K}[x,y]/\mathcal{I}(V) \cong \mathbb{K}[t]\\ \mathbb{K}[W] &:= \mathbb{K}[x,y]/\mathcal{I}(W) \cong \mathbb{K}[t,t^{-1}]\end{aligned}$$

Conclude that the hyperbola $V(xy - 1)$ is not isomorphic to the affine line.

3. Suppose that $\mathbb{K}$ is an infinite field. Show that $f \in \mathbb{K}[x_1, \ldots, x_n]$ defines the zero function $f\colon \mathbb{K}^n \to \mathbb{K}$ if and only if f is the zero polynomial. (Hint: One direction is easy, and for the other, consider first the case when $n = 1$ and then use induction.)

4. Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal. Show that the factor ring $\mathbb{K}[x_1, \ldots, x_n]/I$ has nilpotent elements if and only if I is not a radical ideal.

1.4 Projective varieties

Projective space and projective varieties are undoubtedly the most important objects in algebraic geometry. We motivate projective space with an example.

Consider the intersection of the parabola $y = x^2$ in the affine plane $\mathbb{K}^2$ with a line, $\ell := \mathcal{V}(ay + bx + c)$. Solving these implied equations gives

$$ax^2 + bx + c \;=\; 0 \qquad \text{and} \qquad y \;=\; x^2\,.$$

There are several cases to consider.

(i) $a \neq 0$ and $b^2 - 4ac > 0$. Then ℓ meets the parabola in two distinct real points.

(i$'$) $a \neq 0$ and $b^2 - 4ac < 0$. While ℓ does not appear to meet the parabola, that is because we have drawn the real picture, and ℓ meets it in two complex conjugate points.

When $\mathbb{K}$ is algebraically closed, then cases (i) and (i$'$) coalesce to the case of $a \neq 0$ and $b^2 - 4ac \neq 0$. These two points of intersection are predicted by Bézout's Theorem in the plane (Theorem 2.3.15).

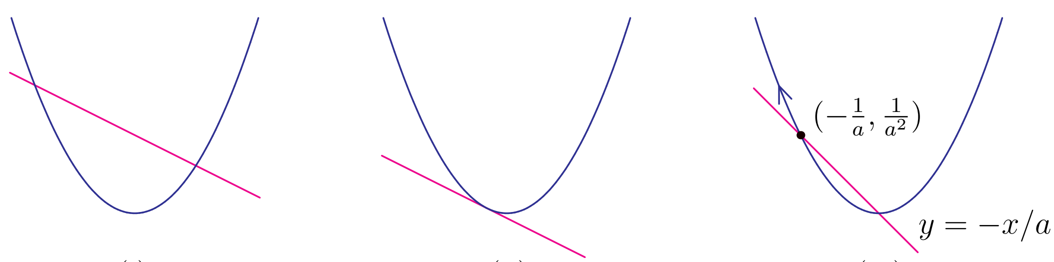
(ii) $a \neq 0$ but $b^2 - 4ac = 0$. Then ℓ is tangent to the parabola and we solve the equations to get

$$a(x - \tfrac{b}{2a})^2 \;=\; 0 \qquad \text{and} \qquad y \;=\; x^2\,.$$

Thus there is one solution, $(\frac{b}{2a}, \frac{b^2}{4a^2})$. As $x = \frac{b}{2a}$ is a root of multiplicity 2 in the first equation, it is reasonable to say that this one solution to our geometric problem occurs with multiplicity 2.

(iii) $a = 0$. There is a single, unique solution, $x = -c/b$ and $y = c^2/b^2$.

Suppose now that $c = 0$ and let $b = 1$. For $a \neq 0$, there are two solutions $(0,0)$ and $(-\frac{1}{a}, \frac{1}{a^2})$. In the limit as $a \to 0$, the second solution disappears off to infinity.



(i) (ii) (iii)

One purpose of projective space is to prevent this last phenomenon from occurring.

Definition 1.4.1. The set of all 1-dimensional linear subspaces of $\mathbb{K}^{n+1}$ is called *n-dimensional projective space* and written $\mathbb{P}^n$ or $\mathbb{P}^n_{\mathbb{K}}$. If V is a finite-dimensional vector space, then $\mathbb{P}(V)$ is the set of all 1-dimensional linear subspaces of V. Note that $\mathbb{P}(V) \simeq \mathbb{P}^{\dim V-1}$, but there are no preferred coordinates for $\mathbb{P}(V)$.

Example 1.4.2. The projective line $\mathbb{P}^1$ is the set of lines through the origin in $\mathbb{K}^2$. When $\mathbb{K} = \mathbb{R}$, we see that the line $x = ay$ through the origin intersects the circle $\mathcal{V}(x^2 + (y-1)^2 - 1)$ in the origin and in the point $(2a/(1+a^2),\, 2/(1+a^2))$, as shown in Figure 1.4. Identifying the x-axis with the origin and the lines $x = ay$ with this point of intersection gives a one-to-one map from $\mathbb{P}^1_{\mathbb{R}}$ to the circle, where the origin becomes the point at infinity.

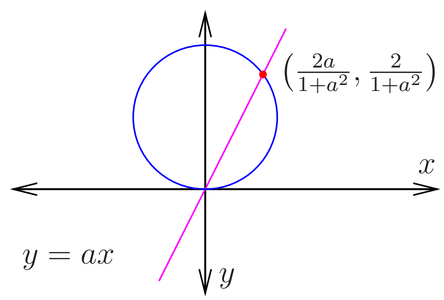


Figure 1.4: Lines through the origin meet the circle in a second point.

This definition of $\mathbb{P}^n$ leads to a system of global homogeneous coordinates for $\mathbb{P}^n$. We may represent a point, ℓ, of $\mathbb{P}^n$ by the coordinates $[a_0, a_1, \ldots, a_n]$ of any non-zero vector lying on the one-dimensional linear subspace $\ell \subset \mathbb{K}^{n+1}$. These coordinates are not unique. If $\lambda \neq 0$, then $[a_0, a_1, \ldots, a_n]$ and $[\lambda a_0, \lambda a_1, \ldots, \lambda a_n]$ both represent the same point. This non-uniqueness is the reason that we use rectangular brackets $[\ldots]$ in our notation for these *homogeneous coordinates*. Some authors prefer the notation $[a_0 : a_1 : \cdots : a_n]$.

Example 1.4.3. When $\mathbb{K} = \mathbb{R}$, note that a 1-dimensional subspace of $\mathbb{R}^{n+1}$ meets the sphere S^n in two antipodal points, v and $-v$. This identifies real projective space $\mathbb{P}^n_{\mathbb{R}}$ with the quotient $S^n/\{\pm 1\}$, showing that $\mathbb{P}^n_{\mathbb{R}}$ is a compact manifold in the usual topology.

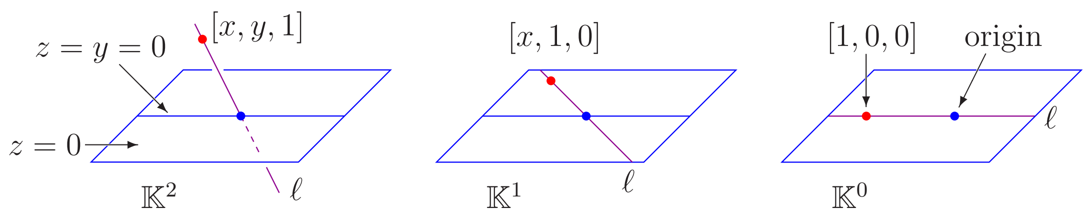
Suppose that $\mathbb{K} = \mathbb{C}$. Given a point $a \in \mathbb{P}^n_{\mathbb{C}}$, after scaling, we may assume that $|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2 = 1$. Identifying $\mathbb{C}$ with $\mathbb{R}^2$, this is the set of points a on the $2n+1$-sphere $S^{2n+1} \subset \mathbb{R}^{2n+2}$. If $[a_0, \ldots, a_n] = [b_0, \ldots, b_n]$ with $a, b \in S^{2n+1}$, then there is some $\zeta \in S^1$, the unit circle in $\mathbb{C}$, such that $a_i = \zeta b_i$. This identifies $\mathbb{P}^n_{\mathbb{C}}$ with the quotient of S^{2n+1}/S^1, showing that $\mathbb{P}^n_{\mathbb{C}}$ is a compact manifold. Since $\mathbb{P}^n_{\mathbb{R}} \subset \mathbb{P}^n_{\mathbb{C}}$, we again see that $\mathbb{P}^n_{\mathbb{R}}$ is compact.

Homogeneous coordinates of a point are not unique. Uniqueness may be restored, but at the price of non-uniformity. Let $A_i \subset \mathbb{P}^n$ be the set of points $[a_0, a_1, \ldots, a_n]$ in projective space $\mathbb{P}^n$ with $a_i \neq 0$, but $a_{i+1} = \cdots = a_n = 0$. Given a point $a \in A_i$, we may divide by its ith coordinate to get a representative of the form $[a_0, \ldots, a_{i-1}, 1, 0, \ldots, 0]$. These i numbers $(a_0, \ldots, a_{i-1})$ provide coordinates for A_i, identifying it with the affine space $\mathbb{K}^i$. This decomposes projective space $\mathbb{P}^n$ into a disjoint union of $n+1$ affine spaces

$$\mathbb{P}^n \;=\; \mathbb{K}^n \sqcup \cdots \sqcup \mathbb{K}^1 \sqcup \mathbb{K}^0 \,.$$

When a variety admits a decomposition as a disjoint union of affine spaces, we say that it is *paved by affine spaces*. Many important varieties admit such a decomposition.

It is instructive to look at this closely for $\mathbb{P}^2$. Below, we show the possible positions of a one-dimensional linear subspace $\ell \subset \mathbb{K}^3$ with respect to the x, y-plane $z = 0$, the x-axis $z = y = 0$, and the origin in $\mathbb{K}^3$.



There is also a scheme for local coordinates on projective space.

1. For $i = 0, \ldots, n$, let U_i be the set of points $a \in \mathbb{P}^n$ in projective space whose ith coordinate is non-zero. Dividing by this ith coordinate, we obtain a representative of the point having the form

$$[a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n]\,.$$

   The n coordinates $(a_0, \ldots, a_{i-1},\ a_{i+1}, \ldots, a_n)$ determine this point, identifying U_i with affine n-space, $\mathbb{K}^n$. Every point of $\mathbb{P}^n$ lies in some U_i,

$$\mathbb{P}^n \;=\; U_0 \cup U_1 \cup \cdots \cup U_n\,.$$

   When $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, these U_i are coordinate charts for $\mathbb{P}^n$ as a manifold.

   For any field $\mathbb{K}$, these affine sets U_i provide coordinate charts for $\mathbb{P}^n$.

2. We give a coordinate-free description of these affine charts. Let $\Lambda\colon \mathbb{K}^{n+1} \to \mathbb{K}$ be a linear map, and let $H \subset \mathbb{K}^{n+1}$ be the set of points x where $\Lambda(x) = 1$. Then $H \simeq \mathbb{K}^n$, and the map

$$H \ni x \;\longmapsto\; [x] \in \mathbb{P}^n$$

   identifies H with the complement $U_\Lambda = \mathbb{P}^n - \mathcal{V}(\Lambda)$ of the points where Λ vanishes.

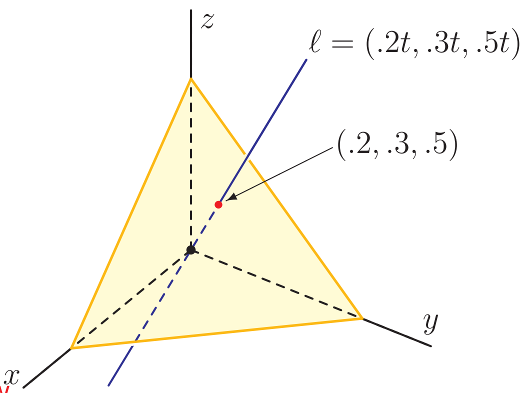
Example 1.4.4 (Probability simplex)**.** This more general description of affine charts leads to the beginning of an important application of algebraic geometry to statistics.

Here $\mathbb{K} = \mathbb{R}$, the real numbers and we set $\Lambda(x) := x_0 + \cdots + x_n$. If we consider those points x where $\Lambda(x) = 1$ which have positive coordinates, we obtain the *probability simplex*

$$\Delta := \{(p_0, p_1, \ldots, p_n) \in \mathbb{R}^{n+1}_+ \mid p_0 + p_1 + \cdots + p_n = 1\},$$

where $\mathbb{R}^{n+1}_+$ is the *positive orthant*, the points of $\mathbb{R}^{n+1}$ with nonnegative coordinates. Here p_i represents the probability of an event i occurring, and the condition $p_0 + \cdots + p_n = 1$ reflects that every event does occur.

Here is a picture when $n = 2$.



Rotate the picture and redraw.

We wish to extend the definitions and structures of affine algebraic varieties to projective space. One problem arises immediately: given a polynomial $f \in \mathbb{K}[x_0, \ldots, x_n]$ and a point $a \in \mathbb{P}^n$, we cannot in general define $f(a) \in \mathbb{K}$. To see why this is the case, for each non negative integer d, let f_d be the sum of the terms of f of degree d.[1] We call f_d the dth *homogeneous component* of f. If $[a_0, \ldots, a_n]$ and $[\lambda a_0, \ldots, \lambda a_n]$ are two representatives of $a \in \mathbb{P}^n$, and f has degree m, then

$$f(\lambda a_0, \ldots, \lambda a_n) = f_0(a_0, \ldots, a_n) + \lambda f_1(a_0, \ldots, a_n) + \cdots + \lambda^m f_m(a_0, \ldots, a_n), \tag{1.6}$$

since we can factor λ^d from every monomial $(\lambda x)^\alpha$ of degree d. Thus $f(a)$ is a well-defined number only if the polynomial (1.6) in λ is constant. That is, if and only if

$$f_i(a_0, \ldots, a_n) = 0 \qquad i = 1, \ldots, \deg(f).$$

In particular, a polynomial f vanishes at a point $a \in \mathbb{P}^n$ if and only if every homogeneous component f_d of f vanishes at a. A polynomial f is *homogeneous* of degree d when $f = f_d$. We also use the term *homogeneous form* for a homogeneous polynomial.

Definition 1.4.5. Let $f_1, \ldots, f_m \in \mathbb{K}[x_0, \ldots, x_n]$ be homogeneous polynomials. These define a *projective variety*

$$\mathcal{V}(f_1, \ldots, f_m) := \{a \in \mathbb{P}^n \mid f_i(a) = 0,\ i = 1, \ldots, m\}.$$

[1] Define degree!

An ideal $I \subset \mathbb{K}[x_0, \ldots, x_n]$ is *homogeneous* if whenever $f \in I$ then all homogeneous components of f lie in I. Thus projective varieties are defined by homogeneous ideals. Given a subset $Z \subset \mathbb{P}^n$ of projective space, its ideal is the collection of polynomials which vanish on Z,

$$\mathcal{I}(Z) \ := \ \{f \in \mathbb{K}[x_0, x_1, \ldots, x_n] \mid f(z) = 0 \text{ for all } z \in Z\}\,.$$

In the exercises, you are asked to show that this ideal is homogeneous.

It is often convenient to work in an affine space when treating projective varieties. The *(affine) cone CZ* $\subset \mathbb{K}^{n+1}$ over a subset Z of projective space $\mathbb{P}^n$ is the union of the one-dimensional linear subspaces $\ell \subset \mathbb{K}^{n+1}$ corresponding to points of Z. Then the ideal $\mathcal{I}(X)$ of a projective variety X is equal to the ideal $\mathcal{I}(CX)$ of the affine cone over X.

Example 1.4.6. Let $\Lambda := a_0x_0 + a_1x_1 + \cdots + a_nx_n$ be a linear form. Then $\mathcal{V}(\Lambda)$ is a *hyperplane*. Let $V \subset \mathbb{K}^{n+1}$ be the kernel of Λ which is an n-dimensional linear subspace. It is also the affine variety defined by Λ. We have $\mathcal{V}(\Lambda) = \mathbb{P}(V)$.

The weak Nullstellensatz does not hold for projective space, as $\mathcal{V}(x_0, x_1, \ldots, x_n) = \emptyset$. We call this ideal, $\mathfrak{m}_0 := \langle x_0, x_1, \ldots, x_n\rangle$, the *irrelevant ideal*. It plays a special role in the projective algebraic-geometric dictionary.

Theorem 1.4.7 (Projective Algebraic-Geometric Dictionary)**.** *Over any field $\mathbb{K}$, the maps $\mathcal{V}$ and $\mathcal{I}$ give an inclusion reversing correspondence*

$$\left\{\begin{array}{c}\textit{Radical homogeneous ideals } I \textit{ of}\\ \mathbb{K}[x_0, \ldots, x_n] \textit{ properly contained in } \mathfrak{m}_0\end{array}\right\} \quad \underset{\mathcal{I}}{\overset{\mathcal{V}}{\rightleftarrows}} \quad \{\textit{Subvarieties } X \textit{ of } \mathbb{P}^n\}$$

with $\mathcal{V}(\mathcal{I}(X)) = X$. When $\mathbb{K}$ is algebraically closed, the maps $\mathcal{V}$ and $\mathcal{I}$ are inverses, and this correspondence is a bijection.

We can deduce this from the algebraic-geometric dictionary for affine space (Corollary 1.2.10), if we replace a subvariety X of projective space by its affine cone CX.

If we relax the condition that an ideal be radical, then the corresponding geometric objects are *projective schemes*. This comes at a price, for many homogeneous ideals will define the same projective scheme. This non-uniqueness comes from the irrelevant ideal, $\mathfrak{m}_0$. Recall the construction of colon ideals. Let I and J be ideals. Then the *colon ideal* (or *ideal quotient* of I by J) is

$$(I : J) \ := \ \{f \mid fJ \subset I\}\,.$$

An ideal $I \subset \mathbb{K}[x_0, x_1, \ldots, x_n]$ is *saturated* if

$$I \ = \ (I : \mathfrak{m}_0) \ := \ \{f \mid x_if \in I \text{ for } i = 0, 1, \ldots, n\}\,.$$

The reason for this definition is that I and $(I : \mathfrak{m}_0)$ define the same projective scheme.

Given a projective variety $X \subset \mathbb{P}^n$, we may consider its intersection with any affine open subset $U_i = \{x \in \mathbb{P}^n \mid x_i \neq 0\}$. For simplicity of notation, we will work with $U_0 = \{[1, x_1, \ldots, x_n] \mid (x_1, \ldots, x_n) \in \mathbb{K}^n\} \simeq \mathbb{K}^n$. Then

$$X \cap U_0 \;=\; \{a \in U_0 \mid f(a) = 0 \text{ for all } f \in \mathcal{I}(X)\}\,.$$

and

$$\mathcal{I}(X \cap U_0) \;=\; \{f(1, x_1, \ldots, x_n) \mid f \in \mathcal{I}(X)\}\,.$$

We call the polynomial $f(1, x_1, \ldots, x_n)$ the *dehomogenization* of the homogeneous polynomial f. This shows that the ideal of $X \cap U_0$ is obtained by dehomogenizing the polynomials in the ideal of X. Note that f and $x_0^m f$ both dehomogenize to the same polynomial.

Conversely, given an affine subvariety $Y \subset U_0$, we have its Zariski closure[2] $\overline{Y} := \mathcal{V}(\mathcal{I}(Y)) \subset \mathbb{P}^n$. The relation between the ideal of the affine variety Y and homogeneous ideal of its closure $\overline{Y}$ is through homogenization.

$$\begin{aligned}
\mathcal{I}(\overline{Y}) \;&=\; \{f \in \mathbb{K}[x_0, \ldots, x_n] \mid f|_Y = 0\} \\
&=\; \{f \in \mathbb{K}[x_0, \ldots, x_n] \mid f(1, x_1, \ldots, x_n) \in \mathcal{I}(Y) \subset \mathbb{K}[x_1, \ldots, x_n]\} \\
&=\; \{x_0^{\deg(g)+m} g(\tfrac{x_1}{x_0}, \ldots, \tfrac{x_n}{x_0}) \mid g \in \mathcal{I}(Y),\ m \geq 0\}\,.
\end{aligned}$$

The point of this is that every projective variety X is naturally a union of affine varieties

$$X \;=\; \bigcup_{i=0}^{n} \bigl(X \cap U_i\bigr)\,.$$

This gives a relationship between varieties and manifolds: Affine varieties are to varieties what open subsets of $\mathbb{R}^n$ are to manifolds.

Could define quasi-projective varieties

[2]Use this notion earlier for closures of maps, but mention it is developed in Chapter 3.

1.5 Coordinate rings and maps of projective varieties

Given a projective variety $X \subset \mathbb{P}^n$, its *homogeneous coordinate ring* $\mathbb{K}[X]$ is the quotient

$$\mathbb{K}[X] := \mathbb{K}[x_0, x_1, \ldots, x_n]/\mathcal{I}(X).$$

If we set $\mathbb{K}[X]_d$ to be the images of all degree d homogeneous polynomials, $\mathbb{K}[x_0, \ldots, x_n]_d$, then this ring is graded,

$$\mathbb{K}[X] = \bigoplus_{d \geq 0} \mathbb{K}[X]_d,$$

where if $f \in \mathbb{K}[X]_d$ and $g \in \mathbb{K}[X]_e$, then $fg \in \mathbb{K}[X]_{d+e}$. More concretely, we have

$$\mathbb{K}[X]_d = \mathbb{K}[x_0, \ldots, x_n]_d/\mathcal{I}(X)_d,$$

where $\mathcal{I}(X)_d = \mathcal{I}(X) \cap \mathbb{K}[x_0, \ldots, x_n]_d$.

This differs from the coordinate ring of an affine variety as its elements are not functions on X, as we already observed that, apart from constant polynomials, elements of $\mathbb{K}[x_0, \ldots, x_n]$ do not give functions on $\mathbb{P}^n$.

Maps of projective varieities need to be treated much more carefully

However, given two homogeneous polynomials f and g which have the same degree, d, the quotient f/g does give a well-defined function, at least on $\mathbb{P}^n - \mathcal{V}(g)$. Indeed, if $[a_0, \ldots, a_n]$ and $[\lambda a_0, \ldots, \lambda a_n]$ are two representatives of the point $a \in \mathbb{P}^n$ and $g(a) \neq 0$, then

$$\frac{f(\lambda a_0, \ldots, \lambda a_n)}{g(\lambda a_0, \ldots, \lambda a_n)} = \frac{\lambda^d f(a_0, \ldots, a_n)}{\lambda^d g(a_0, \ldots, a_n)} = \frac{f(a_0, \ldots, a_n)}{g(a_0, \ldots, a_n)}.$$

It follows that if $f, g \in \mathbb{K}[X]$ with $g \neq 0$, then the quotient f/g gives a well-defined function on $X - \mathcal{V}(g)$.

More generally, let $f_0, f_1, \ldots, f_m \in \mathbb{K}[X]$ be elements of the same degree with at least one f_i non-zero on X. These define a *rational map*

$$\begin{array}{rccl} \varphi : & X & \dashrightarrow & \mathbb{P}^m \\ & x & \longmapsto & [f_0(x), f_1(x), \ldots, f_m(x)]. \end{array}$$

This is defined at least on the set $X - \mathcal{V}(f_0, \ldots, f_m)$. A second list $g_0, \ldots, g_m \in \mathbb{K}[X]$ of elements of the same degree (possible different from the degrees of the f_i) defines the same rational map if we have

$$\operatorname{rank}\begin{bmatrix} f_0 & f_1 & \cdots & f_m \\ g_0 & g_1 & \cdots & g_m \end{bmatrix} = 1 \qquad \text{i.e.} \quad f_i g_j - f_j g_i \in \mathcal{I}(X) \ \text{ for } i \neq j.$$

The map φ is regular at a point $x \in X$ if there is some system of representatives $f_0, \ldots, f_m$ for the map φ for which $x \notin \mathcal{V}(f_0, \ldots, f_m)$. The set of such points is an open subset of X called the *domain of regularity* of φ. The map φ is *regular* if it is regular at all points of X. The *base locus* of a rational map $\varphi \colon X \dashrightarrow Y$ is the set of points of X at which φ is not regular.

Example 1.5.1. An important example of a rational map is a linear projection. Let $\Lambda_0, \Lambda_1, \ldots, \Lambda_m$ be linear forms. These give a rational map φ which is defined at points of $\mathbb{P}^n - E$, where E is the common zero locus of the linear forms $\Lambda_0, \ldots, \Lambda_m$, that is $E = \mathbb{P}(\text{kernel}(L))$, where L is the matrix whose columns are the Λ_i.

The identification of $\mathbb{P}^1$ with the points on the circle $\mathcal{V}(x^2 + (y-1)^2 - 1) \subset \mathbb{K}^2$ from Example 1.4.2 is an example of a linear projection. Let $X := \mathcal{V}(x^2 + (y-z)^2 - z^2)$ be the plane conic which contains the point $[0, 0, 1]$. The identification of Example 1.4.2 was the map

$$\mathbb{P}^1 \ \ni \ [a, b] \ \longmapsto \ [2ab, 2a^2, a^2 + b^2] \ \in \ X\,.$$

Its inverse is the linear projection $[x, y, z] \mapsto [x, y]$.

Figure 1.5 shows another linear projection. Let C be the cubic space curve with parametrization $[1, t, t^2, 2t^3 - 2t]$ and $\pi\colon \mathbb{P}^3 \dashrightarrow L \simeq \mathbb{P}^1$ the linear projection defined by the last two coordinates, $\pi\colon [x_0, x_1, x_2, x_3] \mapsto [x_3, x_4]$. We have drawn the image $\mathbb{P}^1$ in the picture to illustrate that the inverse image of a linear projection is a linear section of the variety (after removing the base locus). The center of projection is a line, E, which meets

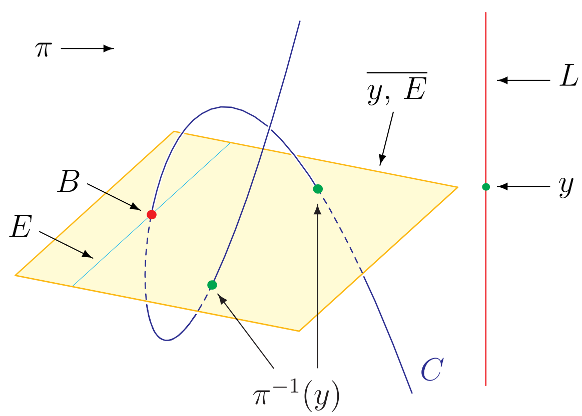


Figure 1.5: A linear projection π with center E.

the curve in a point, B.

Projective varieties $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ are *isomorphic* if we have regular maps $\varphi\colon X \to Y$ and $\psi\colon Y \to X$ for which the compositions $\psi \circ \varphi$ and $\varphi \circ \psi$ are the identity maps on X and Y, respectively.

Exercises

1. A transition function $\varphi_{i,j}$ expresses how to change from the local coordinates from U_i of a point $p \in U_i \cap U_j$ to the local coordinates from U_j. Write down the transition functions for $\mathbb{P}^n$ provided by the affine charts $U_0, \ldots, U_n$.

2. Show that an ideal I is homogeneous if and only if it is generated by homogeneous polynomials.

3. Let $Z \subset \mathbb{P}^n$. Show that $\mathcal{I}(Z)$ is a homogeneous ideal.

4. Show that a radical homogeneous ideal is saturated.

5. Show that the homogeneous ideal $\mathcal{I}(Z)$ of a subset $Z \subset \mathbb{P}^n$ is equal to the ideal $\mathcal{I}(CZ)$ of the affine cone over Z.

6. Verify the claim in the text concerning the relation between the ideal of an affine subvariety $Y \subset U_0$ and of its Zariski closure $\overline{Y} \subset \mathbb{P}^n$:

$$\mathcal{I}(\overline{Y}) \;=\; \{x_0^{\deg(g)+m} g(\tfrac{x_1}{x_0}, \dots, \tfrac{x_n}{x_0}) \mid g \in \mathcal{I}(Y) \subset \mathbb{K}[x_1, \dots, x_n],\ m \geq 0\}\,.$$

7. Let $X \subset \mathbb{P}^n$ be a projective variety and suppose that $f, g \in \mathbb{K}[X]$ are homogeneous forms of the same degree with $g \neq 0$. Show that the quotient f/g gives a well-defined function on $X - \mathcal{V}(g)$.

8. Show that if I is a homogeneous ideal and J is its *saturation*,

$$J \;=\; \bigcup_{d \geq 0} (I : \mathfrak{m}_0^d)\,,$$

then there is some integer N such that

$$J_d \;=\; I_d \qquad \text{for} \quad d \geq N\,.$$

9. Verify the claim in the text that if $X \subset \mathbb{P}^n$ is a projective variety, then its homogeneous coordinate ring is graded with

$$\mathbb{K}[X]_d \;=\; \mathbb{K}[x_0, \dots, x_n]_d / \mathcal{I}(X)_d\,.$$

1.6 Notes

Most of the material in this chapter is standard material within courses of algebraic geometry or related courses. User-friendly, introductory texts to these topics include the books of Beltrametti, Carletti, Gallarati, and Monti Bragadin [5], Cox, Little, O'Shea [20], Holme [40], Hulek [42], Perrin [67], Smith, Kahanpää, Kekäläinen, and Traves [85]. Advanced, in-depth treatments from the viewpoint of modern, abstract algebraic geometry can be found in the books of Eisenbud [25], Harris [35], Hartshorne [36], and Shafarevich [84].