DERIVATIVES OF MULTIVARIATE FUNCTIONS AND THE CONTINUITY OF MIXED PARTIAL DERIVATIVES

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1. The Derivative

We will briefly review the ordinary derivative of one variable calculus, then try to generalize this to the case of multivariate functions.

1.1. **The single variable derivative.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $a \in \mathbb{R}$ be a real number. Recall that f is differentiable at a with derivative $Df(a)$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = Df(a).$$

This may be rewritten in the following manner:

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - hDf(a)}{h} = 0.$$

Or, replacing h by its absolute value, $|h|$, in the denominator,

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - hDf(a)}{|h|} = 0. \qquad (*)$$

Note that $Df(a)$ is the unique number (if such a number exists) which satisfies $(*)$.

1.2. **Multivariate functions: The case $f : \mathbb{R} \to \mathbb{R}^n$.** Now suppose that $f : \mathbb{R} \to \mathbb{R}^n$. Then f is an n-tuple of functions $(f_1, \ldots, f_n)$ where each $f_j : \mathbb{R} \to \mathbb{R}$. It is reasonable to define the derivative of f at $a \in \mathbb{R}$ to be the n-tuple $Df(a) = (Df_1(a), \ldots, Df_n(a))$. This satisfies the analog of $(*)$:

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - hDf(a)}{|h|} = 0, \qquad (**)$$

as a limit of vectors in $\mathbb{R}^n$. Here, if $g : \mathbb{R} \to \mathbb{R}^n$ is a vector valued function and $L \in \mathbb{R}^n$, then $\lim_{t \to 0} g(t) = L$ means that for every $1 \leq j \leq n$ we have the limit of ordinary functions $\lim_{t \to 0} g_j(t) = L_j$, where g_j is the j^{th} component of g, and L_j the j^{th} component of L.

1.3. **Multivariate functions: The case $f : \mathbb{R}^n \to \mathbb{R}$.** Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. What should be the derivative of f at a? We will reason by analogy with $(**)$. In $(**)$, $f(a)$ is a vector, $|h|$ is the magnitude of h, and $h \mapsto h \cdot Df(a)$ defines a linear map is a linear map from $\mathbb{R} \to \mathbb{R}^n$.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we let $||x||$ denote the length of the vector x, that is, $(x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$. We use $||\cdot||$ instead of $|\cdot|$ to differentiate between the multivariate and single variable situations.

Definition: Let $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. Then the derivative of f at a is the linear map $Df(a) : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Df(a)(h)}{||h||} = 0.$$

Here for a function $g : \mathbb{R}^n \to \mathbb{R}$, $\lim\limits_{h \to 0} g(h) = 0$ means that $\forall \epsilon > 0\ \exists \delta > 0$ such that if $||h|| < \delta$, then $|g(h)| < \epsilon$.

2. Partial Derivatives

A linear map $M : \mathbb{R}^n \to \mathbb{R}$ determines a vector $(m_1, \ldots, m_n) \in \mathbb{R}^n$ such that if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $M(x) = m_1 x_1 + \cdots + m_n x_n$. Likewise a vector in $\mathbb{R}^n$ determines a (unique) linear map. We call the components of this vector the *components* of the linear map.

2.1. **Definition.** Let $f : \mathbb{R}^n \to \mathbb{R}$. The components of the map $Df(a)$ are called the *partial derivatives* of f at a, and the j^{th} component is written $\dfrac{\partial f}{\partial x_j}(a)$ or $D_j f(a)$.

We show how to compute these partial derivatives. Let e_j be the vector with a 1 in the j^{th} place and 0's elsewhere, $(0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0)$.

2.2. **Proposition.** $\dfrac{\partial f}{\partial x_j}(a) = \lim\limits_{t \to 0} \dfrac{f(a + te_j) - f(a)}{t}$.

In simple terms, we compute $D_j f(a)$ by treating the variables $x_1, \ldots, \widehat{x_j}, \ldots, x_n$[1] as constants and 'taking the derivative with respect to the x_j variable only'.

Proof: We know that

$$\lim_{h \to 0} \frac{f(a + te_j) - f(a) - Df(a)(te_j)}{||te_j||} = 0.$$

Now, $Df(a)(te_j) = tD_j f(a)$ and $||te_j|| = |t|$, so

$$\lim_{h \to 0} \frac{f(a + te_j) - f(a) - tD_j f(a)}{|t|} = 0. \qquad (*)$$

[1]We often the use the shorthand $x_1, \ldots, \widehat{x_j}, \ldots, x_n$ to indicate that x_j is omitted from the list: $x_1, \ldots, x_{j-1},\ x_{j+1}, \ldots, x_n$

Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(t) = f(a + te_j)$. Then by $(*)$, $Dg(0) = D_j f(a)$. But

$$Dg(0) = \lim_{t\to 0} \frac{g(t) - g(0)}{t} = \lim_{t\to 0} \frac{f(a + te_j) - f(a)}{t}. \quad \blacksquare$$

3. The Continuity of Mixed Partial Derivatives

3.1. **Theorem.** *Let $\mathcal{D}$ be an open subset of $\mathbb{R}^2$ and let $f : \mathcal{D} \to \mathbb{R}$ be a differentiable function such that the mixed partial derivatives D_1D_2f and D_2D_1f exists on $\mathcal{D}$, and let $(a, b) \in \mathcal{D}$ be a point where they are both continuous. Then $D_1D_2f(a, b) = D_2D_1f(a, b)$.*

This is usually expressed as $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

Proof: Our plan is to show that for all $\epsilon > 0$, $|D_1D_2f(a, b) - D_2D_1f(a, b)| < \epsilon$.

Let $\epsilon > 0$. Because both D_1D_2f and D_2D_1f are continuous at (a, b), there exists $\delta > 0$ such that if $||(a, b) - (x, y)|| < \delta$, then we have

$$\begin{aligned} |D_1D_2f(a, b) - D_1D_2f(x, y)| &< \epsilon/2 \\ |D_2D_1f(a, b) - D_2D_1f(x, y)| &< \epsilon/2 \end{aligned}$$

Let $0 < h, k \le \delta/\sqrt{2}$. Then if $a < c < a + h$, and $b < d < b + k$, we have $||(a, b) - (c, d)|| < \delta$.

Define

$$\begin{aligned} (*) &= f(a + h, b + k) - f(a + h, b) - f(a, b + h) + f(a, b) \\ G(y) &= f(a + h, y) - f(a, y) \\ F(x) &= f(x, b + k) - f(x, b) \end{aligned}$$

Then

$$F(a + h) - F(a) = (*) = G(b + k) - G(b).$$

F is continuous on $[a, a+h]$ and differentiable on $(a, a+h)$ and G is continuous on $[b, b+k]$ and differentiable on $(b, b + k)$. By Lagrange's Theorem (Mean Value Theorem), there exists c, d' with $a < c < a + h$ and $b < d' < b + k$ such that

$$hDF(c) = (*) = kDG(d').$$

But

$$\begin{aligned} DF(c) &= D_1f(c, b + k) - D_1f(c, b) \\ DG(d') &= D_2f(a + h, d') - D_2f(a, d') \end{aligned}$$

Applying Lagrange's Theorem once again, we see that there exists c', d with $a < c' < a + h$ and $b < d < b + k$ such that

$$DF(c) = kD_2D_1f(c, d) \quad \text{and} \quad DG(d') = hD_1D_2f(c', d')$$

Thus

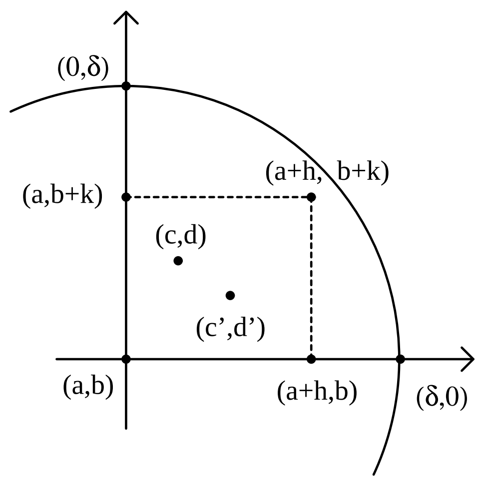
$$hkD_2D_1f(c,d) = (*) = khD_1D_2f(c',d')$$

and so $D_2D_1f(c,d) = D_1D_2f(c',d')$.

We use this to estimate $|D_1D_2f(a,b) - D_2D_1f(a,b)|$.

$$\begin{aligned}|D_1D_2f(a,b) - D_2D_1f(a,b)| \leq \\ &|D_1D_2f(a,b) - D_1D_2f(c',d')| + |D_2D_1f(c,d) - D_2D_1f(a,b)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$ ∎

The figure below should help to visualize the situation of the proof.



4. Exercises

These are to be contemplated.

(1) **Multivariate Derivative** Give a reasonable definition for the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$.

(2) **Multivariate Chain Rule** If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$, we have the composition $g \circ f : \mathbb{R}^n \to \mathbb{R}^p$. What is $D(g \circ f)$?

(3) Define $f : \mathbb{R}^2 \to \mathbb{R}$ as follows: For $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$, set $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$, and $f(0,0) = 0$. Show that f is everywhere continuous, that D_1f, D_2f, $D_{12}f$, and $D_{21}f$ exist everywhere, but that $D_{12}f(0,0) \neq D_{21}f(0,0)$.

Reconcile this with the Theorem of §3.1.