Algorithmic Algebraic Geometry Winter 2019-20
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13 December 2019 — First Homework

I have no idea what is the correct level for this course, so this is experimental.

Think about all of these.
We will discuss them, particularly the odd numbers on Tuesday, December 17.
Hand in the even numbers to Sottile on Friday 20 December.

1. Show that no proper nonempty open subset of $\mathbb{R}^n$ or of $\mathbb{C}^n$ is a variety.
2. Evaluation of a polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$ gives a function on $\mathbb{K}^n$.
   Prove that if $\mathbb{K}$ is infinite, then a polynomial f gives the zero function if and only if f is the zero polynomial.
3. Suppose that X and Y are varieties in $\mathbb{K}^n$. Prove that $X \cap Y$ and $X \cup Y$ are varieties.
4. Show that a single point $a \in \mathbb{K}^n$ is a variety. Prove that any finite set of points in $\mathbb{K}^n$ is a variety.
5. Describe all varieties $X \subset \mathbb{K}$.
6. Let $\mathbb{K}^{m\times n}$ be the set of $m \times n$ matrices over $\mathbb{K}$. (Suppose that $\mathbb{K}$ is infinite for (b).)
   (a) Show that the set of matrices of rank at most r is an algebraic variety.
   (b) Show that the set of matrices of rank exactly r is not an algebraic variety when $r > 0$.
7. Let I be an ideal of $\mathbb{K}[x_1, \ldots, x_n]$. Show that
   $$\sqrt{I} \ := \ \{f \in \mathbb{K}[x_1, \ldots, x_n] \mid f^N \in I, \text{ for some } N \in \mathbb{N}\}$$
   is an ideal, is radical, and is the smallest radical ideal containing I.
8. Let I be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$, where $\mathbb{K}$ is a field. Prove or find counterexamples to the following statements. Make your assumptions clear.
   (a) If $\mathcal{V}(I) = \mathbb{K}^n$ then $I = \langle 0 \rangle$.
   (b) If $\mathcal{V}(I) = \emptyset$ then $I = \mathbb{K}[x_1, \ldots, x_n]$.
9. Let $J = \langle xy, yz, xz \rangle$ be an ideal in $\mathbb{K}[x, y, z]$. Find the generators of $\mathcal{I}(\mathcal{V}(J))$. Show that J cannot be generated by two polynomials in $\mathbb{K}[x, y, z]$. Describe $V(I)$ where $I = \langle xy, xz - yz \rangle$. Show that $\sqrt{I} = J$.
10. Let I be an ideal of $\mathbb{K}[x_1, \ldots, x_n]$. Show that if $\mathbb{K}[x_1, \ldots, x_n]/I$ is a finite dimensional $\mathbb{K}$-vector space then $\mathcal{V}(I)$ is a finite set.