Algorithmic Algebraic Geometry Winter 2019-20
Frank Sottile
Weihnachts, Dezember 2019 **Second Homework**

Due Friday, 10 January 2020.

Please write up your homework neatly. Hand in Numbers 1 and 2, and 5, 9, 10, 11.

1. On Tuesday, 17 December, Sottile ran a script `27lines.m2` in the exercise session. It is available from the course web page or the homework web page. Run this (or the Singular script `27lines.sing`) several times.

   When you do that, comment out the line setting the random seed, so that you count lines on different cubic surfaces. Look up the commands in that file in the corresponding on-line documentation to get some idea of what the computation is doing. Write a paragraph or two about this computation and what you learned.

2. A quartic surface in 4-space is defined by two linearly independent quadratic polynomials $F = G = 0$ in four variables. A line in 4-space may be represented by a point $p = (a, b, c, 0)$ and a direction vector $v = (e, f, g, 1)$ (six coordinates), and parameterized by $\lambda \mapsto p + \lambda v$. Evaluating F and G at $p + \lambda v$ gives two quadratic polynomials in λ. The line $p + \lambda v$ lies on the quartic surface $\mathcal{V}(F, G)$ exactly when the six coefficients of λ in these two quadratic polynomials vanish.

   Modify (including the comments) the file for the 27 lines to compute the number of lines on such a quartic surface in 4-space. How many lines did you find? Explain this computation and what you learned in a paragraph and email the resulting Macaulay2 or Singular script to Sottile and to Thomas Yahl (email to come).

3. Suppose that $\mathbb{K}$ is infinite. Let $V = \mathcal{V}(y - x^2) \subset \mathbb{K}^2$ and $W = \mathcal{V}(xy - 1) \subset \mathbb{K}^2$. Show that

$$\begin{aligned} \mathbb{K}[V] &:= \mathbb{K}[x,y]/\mathcal{I}(V) \cong \mathbb{K}[t]\,, \quad \text{and} \\ \mathbb{K}[W] &:= \mathbb{K}[x,y]/\mathcal{I}(W) \cong \mathbb{K}[t, t^{-1}]\,. \end{aligned}$$

   Conclude that the hyperbola $V(xy - 1)$ is not isomorphic to the affine line.

4. Show that the image of $\mathbb{K}$ under the map $t \mapsto (t^2 - 1, t^3 - t)$ is $\mathcal{V}(y^2 - (x^3 + x^2))$ and its image under $t \mapsto (t^2 + 1, t^3 + t)$ is $\mathcal{V}(y^2 - (x^3 - x^2))$. (Assume that $\mathbb{K}$ is infinite.)

5. Show that $A \mapsto A^{-1}$ is a regular map on $GL_n(\mathbb{K})$. (Clearly state any algebra facts that you use.)

6. Prove the equivalence of the two conditions for an ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ to be a monomial ideal:
   - (i) I is generated by monomials.
   - (ii) If $f \in I$, then every monomial of f lies in I.

7. Show that the radical of a monomial ideal is a monomial ideal, and that a monomial ideal is radical if and only if it has square-free generators. (Square-free means that no variable occurs to a power greater than 1.)
   A monomial ideal is *square-free* when it has square-free generators.

8. Let $\succ$ be a term order. Prove that for any two non-zero polynomials f, g, we have $\mathrm{in}_\succ(fg) = \mathrm{in}_\succ(f)\,\mathrm{in}_\succ(g)$.

9. Show that if an ideal I has a square-free initial ideal, then I is radical. Give an example to show that the converse of this statement is false.

10. Show that each of $\succ_{\mathsf{lex}}$, $\succ_{\mathsf{dlx}}$, and $\succ_{\mathsf{drl}}$, are monomial orders. See Ch. 2, Sect. 2 for the definitions.

11. In $\mathbb{N}$, between any two numbers in the usual order there are only finitely many numbers. Is this necessarily true in $\mathbb{N}^n$ for a monomial order? Is this true for $\succ_{\mathsf{drl}}$? Demonstrate your claims.