Algorithmic Algebraic Geometry Winter 2019-20
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Third Homework

Due Monday, 20 January 2020.

Please write up your homework neatly. Hand in 1,2,4,5,6. Write a computer script for 1,2,4.

1. Let $\succ$ be the degree reverse lexicographic monomial order on $\mathbb{Q}[x,y,z]$ where $x \succ y \succ z$.

   Let $f = xy^2z^2 + xz - yz$ and $G = (x - y^2, y - z^3, z^2 - 1)$.

   (a) Compute $f \mod G$.

   (b) Compute $f \mod G'$, where G' is each of the two cyclic rotations of G.

2. What monomial order gives the smallest (in bit-size) Gröbner basis for the ideal $\langle xy^2 + 2xz + x^4 - y, 7yz + yx^2 + y - z, 3zx + zy^2 + z - 5y\rangle$? This is a contest. The winner gets a Klein Bottle.

3. Suppose that $x_1 \prec x_2 \prec \cdots$. This is the reverse from other problems.

   (a) Use Buchberger's algorithm to compute by hand the reduced Gröbner basis for $\langle x_1 + x_2, x_1x_2\rangle$, and for $\langle x_1 + x_2 + x_3, x_1x_2 + x_1x_3 + x_2x_3, x_1x_2x_3\rangle$ with respect to the degree reverse lexicographic order.

   (b) Do the same for the lexicographic order.

4. Suppose that $x_1 \prec x_2 \prec \cdots$. The elementary symmetric polynomials are defined recursively by $e_k(x_1,\dots,x_n) = 0$ when $k > n$, $e_0 := 1$, $e_1(x_1) := x_1$, and $e_k(x_1,\dots,x_{n+1}) := e_k(x_1,\dots,x_n) + x_{n+1}e_{k-1}(x_1,\dots,x_n)$. Write a program in either `Macaulay2` or `Singular` to compute a Gröbner basis for the ideal $\langle e_k(x_1,\dots,x_n) \mid 1 \leq k \leq n\rangle$ given n, in both lexicographic and degree reverse lexicographic monomial orders, and run this for several values of n. Email your program, and the ones for 1 and 2, to Elise Walker `walkere@math.tamu.edu`. Follow the instructions at `..../~sottile/teaching/20.1/CAG_HW.html`.

   What do you conjecture is the reduced Gröbner basis for this ideal? Can you prove your conjecture?

5. Suppose that $x \succ y \succ z$ in $\mathbb{Q}[x,y,z]$. Yes/no is not sufficient. Prove your assertions.

   (a) Is $\{x^4y^2 - z^5, x^3y^3 - 1, x^2y^4 - 2z\}$ a Gröbner basis for the ideal it generates with respect to the degree reverse lexicographic order?

   (b) The same question, but for $\{x - z^2, y - z^3\}$ with respect to the lexicographic order.

6. Let $f, g \in \mathbb{K}[x_1,\dots,x_n]$ be polynomials with relatively prime initial terms, which we assume are mononials.

   (a) Show that $\mathrm{Spol}(f,g) = -(g - \mathrm{in}(g))f + (f - \mathrm{in}(f))g$. Deduce that the initial monomial of $\mathrm{Spol}(f,g)$ is a multiple of either the initial monomial of f or the initial monomial of g.

   (b) Analyze the steps of the reduction computing $\mathrm{Spol}(f,g) \mod (f,g)$ using the division algorithm to show that this is zero.

7. Let $\succ$ be any monomial order and G be a list of homogeneous polynomials. Show that for any homogeneous polynomial f, its reduction modulo G is also homogeneous.

   Show that Buchberger's algorithm computes a reduced Gröbner basis consisting of homogeneous polynomials. Deduce that the reduced Gröbner basis of a homogeneous ideal consists of homogeneous polynomials.

8. Describe how Buchberger's algorithm behaves when it computes a Gröbner basis from a list of monomials.

9. Suppose that I is an ideal generated by linear forms. Describe its reduced Gröbner basis.

10. Let U be a universal Gröbner basis for an ideal I in $\mathbb{K}[x_1,\dots,x_n]$. Show that for every subset $Y \subset \{x_1,\dots,x_n\}$ the elimination ideal $I \cap \mathbb{K}[Y]$ is generated by $U \cap \mathbb{K}[Y]$.