Algorithmic Algebraic Geometry Winter 2019-20
Frank Sottile
January 2020

Fourth Homework

Due Monday, 27 January 2020.

Please write up your homework neatly. Hand in 1,2,3 to Frank. Write computer scripts for 4, 5, and 6 (except for the challenge on 5). Send the scripts to Thomas: `thomasjyahl@math.tamu.edu`

1. Suppose that $a_1,\dots,a_m$ are the roots of $f(x)$ and $b_1,\dots,b_n$ are the roots of $g(x)$. In the proof of the formula
$$\mathrm{Res}(f,g;x) \;=\; f_0^n g_0^m \prod_{i=1}^{m}\prod_{j=1}^{n}(a_i-b_j) \qquad (1)$$
for the resultant (in which (1) is an identity in $\mathbb{Z}[f_0,g_0,a_1,\dots,a_m,b_1,\dots,b_n]$), it was asserted that the resultant was divisible by each factor (a_i-b_j). Please give a proof of that claim (which may involve the Nullstellensatz and likely uses unique factorization of polynomials).

2. Suppose that the polynomial $g=g_1\cdot g_2$ factors. Show that the resultant also factors, $\mathrm{Res}(f,g;x)=\mathrm{Res}(f,g_1;x)\cdot\mathrm{Res}(f,g_2;x)$.

3. Suppose that $a_1,\dots,a_m$ are the roots of $f(x)$. Prove the equality of the two formulas for the discriminant:
$$\mathrm{disc}_m(f) \;:=\; (-1)^{\binom{m}{2}}\frac{1}{f_0}\mathrm{Res}(f,f') \;=\; f_0^{2m-2}\prod_{i<j}(a_i-a_j)^2\,,$$
Hint: First prove the formula: $f'(a_i)=f_0(a_i-a_1)\cdots\widehat{(a_i-a_i)}\cdots(a_i-a_m)$, where $a_1,\dots,a_m$ are the roots of $f(x)$ and $\widehat{(a_i-a_i)}$ indicates that this term is omitted.

4. Use Gröbner bases to solve the system of equations $f=g=h=0$, where
$$f := 1574y^2-625yx-1234y+334x^4-4317x^3+19471x^2-34708x+19764+45x^2y-244y^3\,,$$
$$g := 45x^2y-305yx-2034y-244y^3-95x^2+655x+264+1414y^2\,, \text{ and}$$
$$h := -33x^2y+197yx+2274y+38x^4-497x^3+2361x^2-4754x+1956+244y^3-1414y^2\,.$$
These polynomials are in a file available on the course web site.

5. Suppose that a,b,c are complex numbers such that
$$a+b+c=3\,,\ \ a^2+b^2+c^2=5\,,\ \text{ and } a^3+b^3+c^3=7\,.$$
Use Gröbner bases to show that $a^4+b^4+c^4=9$. Show that $a^5+b^5+c^5\neq 11$. What are $a^5+b^5+c^5$ and $a^6+b^6+c^6$? Challenge: Can you find a formula for $a^n+b^n+c^n$?

6. Suppose that $f=x^2+y^2+z^2-4$ and $g=4x^2-4y^2+(2z-3)^2-1$. Use a resultant to compute $\pi(\mathcal{V}(f,g))$ where $\pi\colon(x,y,z)\mapsto(x,y)$.

   Suppose that f and g are general polynomials in $\mathbb{C}[x,y,z]$ of degrees a and b, respectively, what do you expect is the degree of $\mathrm{Res}(f,g;z)$? Why ? (This is for discussion.)

7. What happens to the Sylvester matrix $\mathrm{Syl}(f,g)$ if $m\geq n$ and f is replaced by the first remainder in the Euclidean Algorithm applied to f and g?

8. Suppose that G is a reduced Gröbner basis for an ideal I with respect to a monomial order $\prec$. Let $w\in\mathbb{N}^n$ be a weight vector such that for $g\in G$, $\mathrm{in}_w\, g=\mathrm{in}_\prec\, g$.

   (a) Prove that $\mathrm{in}_w\, I=\langle \mathrm{in}_w\, f \mid f\in I\rangle=\mathrm{in}_\prec\, I$. In particular $\mathrm{in}_w\, I$ is a monomial ideal.

   (b) (This is for a discussion.) How do our algorithms, multivariate division, ideal membership, Buchberger, etc. perform using in_w in place of $\prec$ (using them on generators of I)? What if we use G?