Algorithmic Algebraic Geometry Winter 2019-20

Frank Sottile

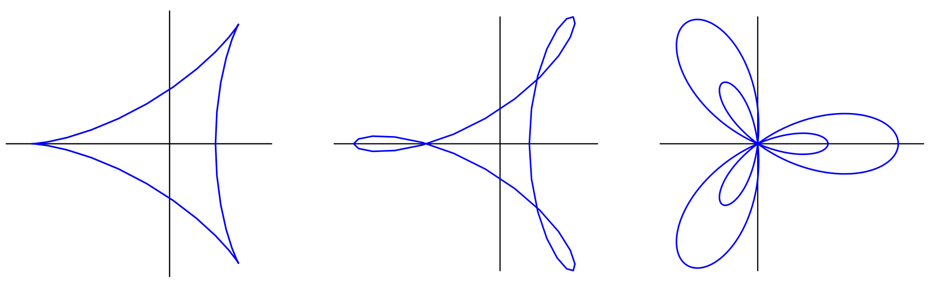
3 February 2020 **Fifth Homework**

Due 11 February 2020.

Please write up your homework neatly. Hand in 1, 2, the answer to the question of 4, your conjecture for 6, and 7 to Frank. Write computer scripts for 3, 4, 5, and 6. Send the scripts 3, 4, and 5 to Elise `walkere@math.tamu.edu` and 6 to Thomas: `thomasjyahl@math.tamu.edu`

The *degree* of a variety $V \subset \mathbb{K}^n$ is the number of points of intersection $V \cap \Lambda$, where Λ is a general linear subspace with $\dim V + \dim \Lambda = n$. It is also the maximum number of points in such an intersection, when the intersection is finite.

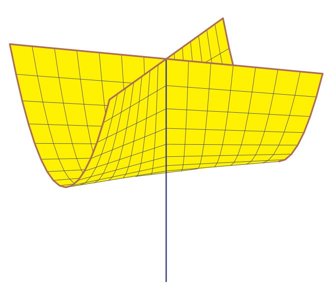
1. Please write up a clear, crisp, and correct solution to number 6 from Homework #3.
2. Using Bézout's Theorem for the number of solutions to a system of polynomial equations, and the definition of degree of a subvariety, prove Bézout's Theorem for the intersection of m general hypersurfaces in $\mathbb{K}^n$.
3. The trigonometric curves parameterized by $(\cos(\theta)-\frac{1}{2}\cos(2\theta), \sin(\theta)+\frac{1}{2}\sin(2\theta)/2)$, $(\cos(\theta)-\frac{2}{3}\cos(2\theta), \sin(\theta)+\frac{2}{3}\sin(2\theta))$, and the polar curve $r = 1+3\cos(3\theta)$ are the cuspidal and trinodal plane quartics, and the rose with three petals, respectively.

   Use elimination to find their implicit equations: Write each as the projection to the (x, y)-plane of an algebraic variety in $\mathbb{K}^4$. Hint: These are images of the circle $c^2 + s^2 = 1$ under maps to the (x, y) plane, where the variables (c, s) correspond to $(\cos(\theta), \sin(\theta))$. The graph of the first is given by the three polynomials

   $$c^2 + s^2 - 1\,,\ x - (c - \tfrac{1}{2}(c^2 - s^2))\,,\ y - (s + sc)\,,$$

   using the identities $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.
4. The Whitney umbrella is the image in $\mathbb{K}^3$ of the map $(u, v) \mapsto (uv, u, v^2)$. Use elimination to find an implicit equation for the Whitney umbrella.

   Which points in $\mathbb{K}^2$ give the handle of the Whitney umbrella?

5. In this and the next exercise, you are asked to use computer experimentation to study the number of solutions to certain structured polynomial systems.

   A polynomial is *multilinear* if all exponents are 0 or 1. For example,

$$3xyz - 17xy + 29xz - 37yz + 43x - 53y + 61z - 71$$

   is a multilinear polynomial in the variables x, y, z. For several small values of n generate n random multilinear polynomials and compute their numbers of common zeroes. You may do this in a nice positive characteristic, say 1009 or 32003.

6. Let $\mathcal{A} \subset \mathbb{N}^n$ be a finite set of integer vectors, which we regard as exponents of monomials in $\mathbb{K}[x_1, \ldots, x_n]$. A polynomial with support $\mathcal{A}$ is a linear combination of monomials whose exponents are from $\mathcal{A}$. For example

$$1 + 3x + 9x^2 + 27y + 81xy + 243xy^2$$

   is a polynomial whose support is the column vectors of $\mathcal{A} = \left(\begin{smallmatrix} 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{smallmatrix}\right)$.

   For many $\mathcal{A}$ with $n = 2$, generate random systems of polynomials with support $\mathcal{A}$ and determine their numbers of isolated solutions. Try to formulate a conjecture about this number of solutions as a function of $\mathcal{A}$.

   This problem will be a major topic of discussion at the exercise session on 11 February.

7. Consider the following iterative algorithm used by the Babylonians to compute $\sqrt{x}$ for $x > 0$. Observe that if $x_i > 0$ and $x_i \neq \sqrt{x}$, then the interval with endpoints x_i and x/x_i contains $\sqrt{x}$ in its interior. Set $x_{i+1} = \frac{1}{2}(x_i + \frac{x}{x_i})$ and repeat. Compare this method of computing square roots to Newton's Method.