Foundations of Mathematics — Math 300 Sections 902, 905

Tuesday 29 September 2020 — Class worksheet

Principle of Mathematical Induction. Let $P(n)$ be a statement for every positive integer n. Suppose that

1. $P(1)$ is true, and
2. for every positive integer k, if $P(k)$ is true, then $P(k+1)$ is true,

then $P(n)$ is true for all positive integers n.

Group Work.

1. Let $n \in \mathbb{N}$. What is the sum of the first n odd integers? Explore this, make a conjecture, and prove it.

2. Explore the divisibility by 3 of positive powers of 4. (E.g. $4^n \mod 3$.) Make a conjecture and prove it.

3. **Polya's Theorem.** *All horses have the same color.*

   We prove that for natural number n if a herd of horses contains n horses, then all the horses in that herd have the same color.

   Base case: The case with just one horse is trivial. If a herd has only one horse, then clearly all horses in that herd have the same color.

   Inductive step: Let $k \in \mathbb{N}$ and assume that in any herd of k horses, all the horses have the same color. Consider now a herd consisting of $k+1$ horses.

   First, exclude one horse from the herd. The remaining k horses form a herd, and by induction, they all have the same color. Likewise, exclude some other horse (not the same one who was first removed) and look only at the other k horses. By the same reasoning, these too, must also be of the same color. Therefore, the first horse that was excluded is of the same color as the non-excluded horses, who in turn are of the same color as the other excluded horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color.

   This completes the proof.

   Discuss the veracity of this proof of Polya's Theorem.