There is a famous formula, Wallis’ Formula, which is shown below. The problems, which follow, lead to a proof of the formula.

\[ \frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{4}{6} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \]

\[ = \lim_{n \to \infty} \left( \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{4}{6} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right) \]

Let \( I_n \) be defined as

\[ I_n = \int_0^{\pi/2} \sin^n x \, dx \quad \text{for } n = 1, 2 \cdots \]

1. Show that the following formulas are valid. Hint: sine reduction formula and induction.

\[ I_{2n} = \int_0^{\pi/2} \sin^{2n} x \, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2} \quad \text{for } n = 1, 2 \cdots \] (1)

\[ I_{2n+1} = \int_0^{\pi/2} \sin^{2n+1} x \, dx = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \quad \text{for } n = 1, 2 \cdots \] (2)

The key to proving the above formulas is the following reduction formula:

\[ I_k = \int_0^{\pi/2} \sin^k x \, dx = \frac{k-1}{k} \int_0^{\pi/2} \sin^{k-2} x \, dx = \frac{k-1}{k} I_{k-2} \quad \text{(3)} \]

This formula is proven using integration by parts:

\[ \int_0^{\pi/2} \sin^k x \, dx = \int_0^{\pi/2} \sin^{k-1} x \sin x \, dx \]

\[ = -\cos x \sin^{k-1} x \bigg|_0^{\pi/2} + \int_0^{\pi/2} (k-1) \sin^{k-2} x \cos^2 x \, dx \]

\[ = \int_0^{\pi/2} (k-1) \sin^{k-2} x \, dx - \int_0^{\pi/2} (k-1) \sin^k x \, dx \]

Solving this equation for \( I_k \) leads to formula (3).

Equation (1) is verified with an induction argument. When \( n = 1 \) we have

\[ I_2 = \int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} \frac{1 - \cos(2x)}{2} \, dx = \frac{\pi}{4}, \]

and the right hand side of (1) with \( n = 1 \) becomes \( \left( \frac{1}{2} \right) \frac{\pi}{2} \), which verifies that (1) is true when \( n = 1 \).
Assume now that \( (1) \) is true for \( n \). The following lines show it is then true for \( n + 1 \).

\[
I_{2(n+1)} = \left( \frac{2(n+1) - 1}{2(n+1)} \right) I_{2n} = \left( \frac{2(n+1) - 1}{2(n+1)} \right) \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \pi}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2}
\]

\[
= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) \pi}{2 \cdot 4 \cdot 6 \cdots (2n)(2n+1)}
\]

The argument to verify equation (2) is exactly the same with a slight difference when \( n = 1 \), which is shown below.

\[
I_3 = \left( \frac{2}{3} \right) I_1 = \left( \frac{2}{3} \right) \int_0^{\pi/2} \sin x \, dx = \frac{2}{3},
\]

and this is exactly the right hand side of (2) when \( n = 1 \).

2. Show that \( \frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2} \).

This is obvious given equation (3).

3. Show that \( I_{2n+2} \leq I_{2n+1} \leq I_{2n} \).

For \( x \in \left[ 0, \frac{\pi}{2} \right] \) we know that \( 0 \leq \sin x \leq 1 \), which means that \( \sin^{2n+2} x \leq \sin^{2n+1} x \leq \sin^{2n} x \). Thus, we have

\[
\int_0^{\pi/2} \sin^{2n+2} \, dx \leq \int_0^{\pi/2} \sin^{2n+1} \, dx \leq \int_0^{\pi/2} \sin^{2n} \, dx \quad \text{or} \quad (4)
\]

\[
I_{2n+2} \leq I_{2n+1} \leq I_{2n} \quad \text{(5)}
\]

4. Show that \( \frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1 \). Show that \( \lim_{n \to \infty} \frac{I_{2n+1}}{I_{2n}} = 1 \).

This is clear from inequality (5), from problem 2, and the squeeze theorem.

5. Show that \( \frac{\pi}{2} = \lim_{n \to \infty} \left( \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots 2n \cdot 2n}{1 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots (2n-1) \cdot (2n+1)} \right) \).

That this limit equals \( \frac{\pi}{2} \), follows from problem 4 and the fact that

\[
\frac{\pi}{2} \left( \frac{I_{2n+1}}{I_{2n}} \right) = \frac{\pi}{2} \left( \frac{\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdots (2n+1)} \cdot \frac{\pi}{2}}{\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdots (2n+1)} \cdot \frac{\pi}{2}} \right)
\]

\[
= \pi \left( \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots 2n \cdot 2n}{1 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots (2n-1) \cdot (2n+1)} \right)
\]

\[
= \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots 2n \cdot 2n}{1 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots (2n-1) \cdot (2n+1)}
\]

\[
= \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}
\]

\[
= \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}
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\[
= \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}
\]

\[
= \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}
\]