Converting the Black-Scholes PDE to The Heat Equation

The Black-Scholes partial differential equation and boundary value problem is

$$L(V) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \;\; 0 \le S, \;\; 0 \le t \le T$$
$$V(S,T) = f(S), \;\; 0 \le S, \qquad V(0,t) = 0, \;\; 0 \le t \le T\,.$$

If V is the price of a call option, then the boundary condition $f(S) = \max(S - E, 0)$, where E denotes the strike price of the call option.

The following change of variables transforms the Black-Scholes boundary value problem into a standard boundary value problem for the heat equation.

$$\begin{aligned} S &= e^x,\, t = T - \frac{2\tau}{\sigma^2}, \\ V(S,t) &= v(x,\tau) = v\left(\ln(S), \frac{\sigma^2}{2}(T-t)\right). \end{aligned}$$

The partial derivatives of V with respect to S and t expressed in terms of partial derivatives of v in terms of x and τ are:

$$\begin{aligned} \frac{\partial V}{\partial t} &= -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} \\ \frac{\partial V}{\partial S} &= \frac{1}{S}\frac{\partial v}{\partial x} \\ \frac{\partial^2 V}{\partial S^2} &= -\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2} \end{aligned}$$

Placing these expressions into the Black-Scholes partial differential equation and simplifying we have

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v\,.$$

Setting $\kappa = 2r/\sigma^2$ and $t = \tau$, the Black-Scholes boundary value problem becomes

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + (\kappa - 1)\frac{\partial v}{\partial x} - \kappa v\,, \;\; -\infty < x < \infty, \;\; 0 \le t \le \frac{\sigma^2}{2}T \\ v(x,0) &= V(e^x, T) = f(e^x)\,, \;\; -\infty < x < \infty \end{aligned}$$

One more change of variables is needed in order to eliminate the last two terms on the right hand side of the last equation. To this end set

$$v\,(x,t) = e^{\alpha x+\beta t} u\,(x,t) = \phi u\,,$$

where we'll pick α and β later. Computing the partials of v in terms of x and t we have

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \beta \phi u + \phi \frac{\partial u}{\partial t} \\
\frac{\partial v}{\partial x} &= \alpha \phi u + \phi \frac{\partial u}{\partial x} \\
\frac{\partial^2 v}{\partial x^2} &= \alpha^2 \phi u + 2\alpha\phi \frac{\partial u}{\partial x} + \phi \frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

Placing these expressions into the partial differential equation which v satisfies, and setting

$$\begin{aligned}
\alpha &= -\frac{1}{2}\,(k-1) = \frac{\sigma^2 - 2r}{2\sigma^2} \\
\beta &= -\frac{1}{4}\,(k+1)^2 = -\left(\frac{\sigma^2+2r}{2\sigma^2}\right)^2 .
\end{aligned}$$

we have

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ -\infty < x < \infty,\ \ 0 \le t \le \frac{\sigma^2}{2} T \tag{1}$$

$$u\,(x,0) = e^{-\alpha x} v\,(x,0) = e^{-\alpha x} f\,(e^x)\,, \quad -\infty < x < \infty \tag{2}$$

If the option is a call option, with strike price E, then $f(x) = \max(x-E,0)$, and

$$u(x,0) = e^{-\alpha x} \max\,(e^x - E, 0)\ .$$

It can be shown that the solution to the heat equation (1) and initial condition (2) is given by the following integral

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u(\xi,0) e^{-\frac{(x-\xi)^2}{4t}}\, d\xi\,.$$

Find the value of an option, whose value at expiration equals $f(S)$, where

$$f(S) = \begin{cases} 0, & S < 1 \\ 3, & 1 \le S \le 2 \\ 0, & S > 3 \end{cases} .$$

$$\begin{aligned}
V(S,0) &= \nu\left(\ln S, \frac{\sigma^2 T}{2}\right) = e^{\alpha \ln S} e^{\beta \frac{\sigma^2 T}{2}} u\left(\ln S, \frac{\sigma^2 T}{2}\right) \\
&= e^{\alpha \ln S} e^{\beta \frac{\sigma^2 T}{2}} \frac{1}{\sqrt{4\pi \frac{\sigma^2 T}{2}}} \int_{-\infty}^{\infty} u(\xi, 0) e^{-\frac{(\ln S - \xi)^2}{4 \frac{\sigma^2 T}{2}}} \, d\xi \\
&= e^{\alpha \ln S} e^{\beta \frac{\sigma^2 T}{2}} \frac{1}{\sqrt{2\pi \sigma^2 T}} \int_{-\infty}^{\infty} e^{-\alpha \xi} f(e^{\xi}) e^{-\frac{(\ln S - \xi)^2}{2\sigma^2 T}} \, d\xi \\
&= e^{\alpha \ln S} e^{\beta \frac{\sigma^2 T}{2}} \frac{3}{\sqrt{2\pi \sigma^2 T}} \int_{0}^{\ln 2} e^{-\alpha \xi} e^{-\frac{(\ln S - \xi)^2}{2\sigma^2 T}} \, d\xi \\
&= e^{\alpha \ln S} e^{\beta \frac{\sigma^2 T}{2}} \frac{3 S^{-\alpha}}{\sqrt{2\pi \sigma^2 T}} e^{\frac{\alpha^2 \sigma^2 T}{2}} \int_{\lambda_1}^{\lambda_2} e^{-\lambda^2/2} \, d\lambda \qquad \begin{cases} \lambda_1 = \frac{\ln(S/2) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \\ \lambda_2 = \frac{\ln S + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \end{cases} \\
&= 3 e^{-\frac{\sigma^2 + 8r}{8} T} \left[N(\lambda_2) - N(\lambda_1)\right] .
\end{aligned}$$