1 Introduction

We have seen how to price European style options on non-dividend paying stocks. In the following paragraphs we discuss how to place a value on an option for a dividend paying stock. 

Two cases are considered. The first assumes that dividends are paid at a constant rate continuously, while the second assumes that there is a single dividend payment during the remaining life of the option.

2 Continuous Dividends

It may seem unreasonable to construct a model in which dividends are paid continuously. However, while that is unreasonable for a single stock, it is not unreasonable for options on indexed funds. For example, if you purchased some shares in a S&P 500 fund, then one could expect to be receiving dividends at many different times in a year.

Suppose that the asset pays dividends at a constant rate $D_y$, which is called the dividend yield. That is, during time $dt$, $D_ySdt$ dividends are received. Assuming the usual stochastic model we have

$$dS = \mu S dt + \sigma S dB - D_y S dt = (\mu - D_y)S dt + \sigma S dB. \tag{1}$$

Proceeding in the same fashion as in the derivation of the Black-Scholes partial differential equation, we construct a portfolio $\Pi = aS - V$, where $V$ is the price of the option and $a$ will be picked so that the value of the portfolio is deterministic.

$$d\Pi = adS - dV + aD_y S dt = adS - \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt - \frac{\partial V}{\partial S} dS + aD_y S dt$$

The term $aD_y S dt$ arises since the stock pays dividends which increases the value of the portfolio by the amount of the dividend. If we pick $a = \frac{\partial V}{\partial S}$, we then have

$$d\Pi = -\left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + aD_y S dt$$

Since the value of the portfolio is risk free we must have

$$d\Pi = r\Pi dt = r(aS - V)dt = -\left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + aD_y S dt$$

This leads to the following equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_y)S \frac{\partial V}{\partial S} - rV = 0 \tag{2}$$

If the following change of dependent variables is made

$$V(S,t) = e^{-D_y(T-t)}V_1(S,t),$$

then the function $V_1$ satisfies the usual Black-Scholes equation with $r$ replaced by $r - D_y$, and has the same final values as $V$. $V_1$ can then be determined by the reduction to the heat equation technique for finding the value of the option.
2.1 Call Option Example

For a call option the above formula becomes

\[ C(S, t) = e^{-D_y(T-t)}C_1(S, t) \]
\[ = e^{-D_y(T-t)}SN(d_{1,1}) - Xe^{-r(T-t)}N(d_{1,2}), \] (3)

where

\[ d_{1,1} = \frac{\ln(S/X) + (r - D_y + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_{1,2} = d_{1,1} - \sigma\sqrt{T-t}. \]

3 One Time Dividend

Here we assume that the underlying asset, typically a stock, will pay a dividend just one time during the life of the option. Let \( d_y \) denote the dividend yield, which will be paid out at time \( t_d \), for \( 0 < t_d < T \). That is, the amount paid out will equal \( d_y S \).

It seems clear that the value of the stock must decrease as soon as the dividend is paid, and that the amount of decrease should equal \( d_y S \). Let \( S(t_d^+) \) and \( S(t_d^-) \) denote the limit from above and below respectively of the value of the stock price at \( t = t_d \). Then we have

\[ S(t_d^+) = (1 - d_y)S(t_d^-) \] (4)

The key to determining the value of the option is the fact that even though the stock price does not vary in a continuous fashion across the dividend payment time, the option must. For if the value of the option has a discontinuous change at time \( t_d \), there will be an arbitrage opportunity. Thus, if \((t, S(t))\) is the time/stock price path, we have

\[ \lim_{t \to t_d^-} V(S(t), t) = \lim_{t \to t_d^+} V(S(t), t) \]
\[ V(S(t_d^-), t_d^-) = V(S(t_d^+), t_d^+) \] (5)
\[ V(S(t_d^-), t_d^-) = V((1 - d_y)S(t_d^-), t_d^+) \] (6)

Therefore, to price an option on a dividend paying asset, solve the Black-Scholes partial differential equation from \( T \) to \( t_d \), use equation (6) to define the value of the option at time \( t_d \), then solve the Black-Scholes equation a second time going from \( t_d \) to \( t = 0 \).
3.1 Call Option Example

The amount of work in solving this problem for a call option is considerably less than the above paragraph indicates. Let \( C_d(S, t) \) denote the value of a European call option on a one time dividend paying asset, and let \( C(S, t; X) \) denote the price of a plain vanilla European call option with strike price \( X \). Both options have the same time to maturity and the same strike price. Then for \( t > t_d \), the two prices must be the same. That is,

\[
C_d(S, t) = C(S, t; X) \quad \text{for} \quad t_d < t \leq T.
\]

At time \( t_d \) equation (6) tells us that

\[
C_d(S, t_d^-) = C_d((1 - d_y)S, t_d^+),
\]

\[
= C((1 - d_y)S, t_d^+; X) \tag{7}
\]

It is easy to see that the function \( C((1 - d_y)S, t; X) \) satisfies the Black-Scholes partial differential equation. Moreover if we check its value at expiration we have

\[
C((1 - d_y)S, T; X) = \max((1 - d_y)S - X, 0) = (1 - d_y) \max(S - (1 - d_y)^{-1}X, 0).
\]

Remember, \( d_y \) is a yield and satisfies \( 0 < d_y < 1 \). From the above equation we realize that \( C((1 - d_y)S, t; X) \) has the same value has a certain percentage of a call option \((1 - d_y)\) with strike price \((1 - d_y)^{-1}X\). Moreover since equation (7) is true for \( t = t_d \), the usual arbitrage argument tells us that this equality must hold for \( 0 \leq t \leq t_d \).

Thus, we must have

\[
C_d(S, t) = (1 - d_y)C(S, t; (1 - d_y)^{-1}X) \quad \text{for} \quad 0 \leq t \leq t_d. \tag{8}
\]

Note that the effect of the dividend decreases the value of the call option, which is reasonable since the holder of the call does not benefit from the dividend.