1 Introduction

We have seen how to price European style options on non-dividend paying stocks. In the following paragraphs we discuss how to place a value on an option for a dividend paying stock.

Two cases are considered. The first assumes that dividends are paid at a constant rate continuously, while the second assumes that there is a single dividend payment during the remaining life of the option.

2 Continuous Dividends

It may seem unreasonable to construct a model in which dividends are paid continuously. However, while that is unreasonable for a single stock, it is not unreasonable for options on indexed funds. For example, if you purchased some shares in a S&P 500 fund, then one could expect to be receiving dividends at many different times in a year.

Suppose that the asset pays dividends at a constant rate D_y, which is called the dividend yield. That is, during time dt, $D_y S\,dt$ dividends are received. Assuming the usual stochastic model we have

$$dS = \mu S\,dt + \sigma S\,dB - D_y S\,dt = (\mu - D_y)S\,dt + \sigma S\,dB. \tag{1}$$

Proceeding in the same fashion as in the derivation of the Black-Scholes partial differential equation, we construct a portfolio $\Pi = aS - V$, where V is the price of the option and a will be picked so that the value of the portfolio is deterministic.

$$d\Pi = adS - dV + aD_y Sdt = adS - \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt - \frac{\partial V}{\partial S} dS + aD_y Sdt$$

The term $aD_y Sdt$ arises since the stock pays dividends which increases the value of the portfolio by the amount of the dividend. If we pick $a = \frac{\partial V}{\partial S}$, we then have

$$d\Pi = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + aD_y Sdt$$

Since the value of the portfolio is risk free we must have

$$d\Pi = r\Pi dt = r(aS - V)dt = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + aD_y Sdt$$

This leads to the following equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_y) S \frac{\partial V}{\partial S} - rV = 0 \tag{2}$$

If the following change of dependent variables is made

$$V(S,t) = e^{-D_y(T-t)} V_1(S,t)\,,$$

then the function V_1 satisfies the usual Black-Scholes equation with r replaced by $r - D_y$, and has the same final values as V. V_1 can then be determined by the reduction to the heat equation technique for finding the value of the option.

2.1 Call Option Example

For a call option the above formula becomes

$$\begin{aligned} C(S,t) &= e^{-D_y(T-t)}C_1(S,t) \\ &= e^{-D_y(T-t)}SN(d_{1,1}) - Xe^{-r(T-t)}N(d_{1,2})\,, \end{aligned} \tag{3}$$

where

$$d_{1,1} = \frac{\ln(S/X) + (r - D_y + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\,, \quad d_{1,2} = d_{1,1} - \sigma\sqrt{T-t}\,.$$

3 One Time Dividend

Here we assume that the underlying asset, typically a stock, will pay a dividend just one time during the life of the option. Let d_y denote the *dividend yield*, which will be paid out at time t_d, for $0 < t_d < T$. That is, the amount paid out will equal

$$d_y S$$

It seems clear that the value of the stock must decrease as soon as the dividend is paid, and that the amount of decrease should equal $d_y S$. Let $S(t_d^+)$ and $S(t_d^-)$ denote the limit from above and below respectively of the value of the stock price at $t = t_d$. Then we have

$$S(t_d^+) = (1 - d_y)S(t_d^-) \tag{4}$$

The key to determining the value of the option is the fact that even though the stock price does not vary in a continuous fashion across the dividend payment time, the option must. For if the value of the option has a discontinuous change at time t_d, there will be an arbitrage opportunity. Thus, if $(t, S(t))$ is the time/stock price path, we have

$$\lim_{t\to t_d^-} V(S(t),t) = \lim_{t\to t_d^+} V(S(t),t)$$

$$V(S(t_d^-),t_d^-) = V(S(t_d^+),t_d^+) \tag{5}$$

$$V(S(t_d^-),t_d^-) = V((1-d_y)S(t_d^-),t_d^+) \tag{6}$$

Therefore, to price an option on a dividend paying asset, solve the Black-Scholes partial differential equation from T to t_d, use equation (6) to define the value of the option at time t_d, then solve the Black-Scholes equation a second time going from t_d to $t = 0$.

3.1 Call Option Example

The amount of work in solving this problem for a call option is considerably less than the above paragraph indicates. Let $C_d(S,t)$ denote the value of a European call option on a one time dividend paying asset, and let $C(S,t;X)$ denote the price of a plain vanilla European call option with strike price X. Both options have the same time to maturity and the same strike price. Then for $t > t_d$, the two prices must be the same. That is,

$$C_d(S,t) = C(S,t;X) \text{ for } t_d < t \leq T\,.$$

At time t_d equation (6) tells us that

$$\begin{aligned} C_d(S,t_d^-) &= C_d((1-d_y)S,t_d^+) \\ &= C((1-d_y)S,t_d^+;X) \end{aligned} \tag{7}$$

It is easy to see that the function $C((1-d_y)S,t;X)$ satisfies the Black-Scholes partial differential equation. Moreover if we check its value at expiration we have

$$C((1-d_y)S,T;X) = \max((1-d_y)S - X,0) = (1-d_y)\max(S-(1-d_y)^{-1}X,0)\,.$$

Remember, d_y is a yield and satisfies $0 < d_y < 1$. From the above equation we realize that $C((1-d_y)S,t;X)$ has the same value has a certain percentage of a call option $(1-d_y)$ with strike price $(1-d_y)^{-1}X$. Moreover since equation (7) is true for $t = t_d$, the usual arbitrage argument tells us that this equality must hold for $0 \leq t \leq t_d$.

Thus, we must have

$$C_d(S,t) = (1-d_y)C(S,t;(1-d_y)^{-1}X) \text{ for } 0 \leq t \leq t_d\,. \tag{8}$$

Note that the effect of the dividend decreases the value of the call option, which is reasonable since the holder of the call does not benefit from the dividend.