Appendix B

Summation Notation

Throughout the text we have used the notation \( \sum_{j=1}^{n} a_j \) and \( \sum_{j=1}^{n} \sum_{k=1}^{m} a_{jk} \) to indicate finite sums of vectors or numbers. We now explain this notation, and discuss some of its properties.

First, let \( a_j, j = 1, 2, \ldots \) be a sequence of numbers or vectors. We want a compact expression for the sum

\[ a_1 + a_2 + \cdots + a_n \]

We thus define \( \sum_{j=1}^{n} a_j \) inductively by

\[
\begin{align*}
\sum_{j=1}^{1} a_j &= a_1 \\
\sum_{j=1}^{n+1} a_j &= \sum_{j=1}^{n} a_j + a_{n+1}
\end{align*}
\]

Thus, if \( a_1 = 5, a_2 = 4, a_3 = 3, a_4 = 2, a_5 = 1 \), then

\[
\begin{align*}
\sum_{j=1}^{1} a_j &= a_1 = 5 \\
\sum_{j=1}^{2} a_j &= \sum_{j=1}^{1} a_j + a_2 = 5 + 4 = 9 \\
\sum_{j=1}^{3} a_j &= \sum_{j=1}^{2} a_j + a_3 = 9 + 3 = 12
\end{align*}
\]

If we have a doubly indexed sequence \( a_{jk} \), sums over both indices are defined as follows:

\[ \sum_{j=1}^{n} \sum_{k=1}^{m} a_{jk} = \sum_{j=1}^{n} b_j \]
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where \( b_j \) equals \( \sum_{k=1}^m a_{jk} \).

**Theorem B.1.**

a. \( \sum_{j=1}^n (a_j + b_j) = \sum_{j=1}^n a_j + \sum_{j=1}^n b_j \)

b. \( \sum_{j=1}^n ca_j = c \sum_{j=1}^n a_j \)

c. \( \sum_{j=1}^m \sum_{k=1}^n a_{jk} = \sum_{k=1}^m \sum_{j=1}^n a_{jk} \)

**Proof.** We prove a and c by induction and leave b to the reader.

a. \( \sum_{j=1}^1 (a_j + b_j) = a_1 + b_1 = \sum_{j=1}^1 a_j + \sum_{j=1}^1 b_j \). Thus statement \( P_1 \) is verified. We next prove \( P_{n+1} \) assuming \( P_n \).

\[
\sum_{j=1}^{n+1} (a_j + b_j) = \sum_{j=1}^{n} (a_j + b_j) + (a_{n+1} + b_{n+1})
\]
\[
= \sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j + a_{n+1} + b_{n+1}
\]
\[
= \sum_{j=1}^{n} a_j + a_{n+1} + \sum_{j=1}^{n} b_j + b_{n+1}
\]
\[
= \sum_{j=1}^{n+1} a_j + \sum_{j=1}^{n+1} b_j
\]

b. We prove this by induction on \( n \). For \( n = 1 \) we have

\[
\sum_{j=1}^1 \sum_{k=1}^m a_{jk} = \sum_{k=1}^m a_{1k} = \sum_{k=1}^m \sum_{j=1}^1 a_{jk} \quad \text{by (B.1a)}
\]

Now assume that the theorem is true for \( n \) and show it is true for \( n + 1 \).

\[
\sum_{j=1}^{n+1} \sum_{k=1}^m a_{jk} = \sum_{j=1}^{n} \sum_{k=1}^m a_{jk} + \sum_{k=1}^m a_{(n+1)k} \quad \text{by (B.1b)}
\]
\[
= \sum_{k=1}^m \sum_{j=1}^{n} a_{jk} + \sum_{k=1}^m a_{(n+1)k} \quad \text{by induction hypothesis}
\]
\[
= \sum_{k=1}^m \left( \sum_{j=1}^{n} a_{jk} + a_{(n+1)k} \right) \quad \text{by a}
\]
\[
= \sum_{k=1}^m \sum_{j=1}^{n+1} a_{jk} \quad \text{by (B.1b)}
\]
Thus, for all $n$ we have $\sum_{j=1}^{n} \sum_{k=1}^{m} a_{jk} = \sum_{k=1}^{m} \sum_{j=1}^{n} a_{jk}$. \qed