Atmospheric Drag

A body falling in air with velocity $\vec{v}$ is subject to a drag force of magnitude $\hat{\gamma}|\vec{v}|^2$ and direction opposite to that of $\vec{v}$, where $\hat{\gamma}$ is a constant that may be empirically determined. Here we shall study the one-dimensional problem of a body falling directly downward.

Let $x(t)$ be the height of the body above ground at time t. When the body is falling, the force on it is

$$m\,\frac{d^2x}{dt^2} = F = -mg + \hat{\gamma}\left(\frac{dx}{dt}\right)^2.$$

So the equation of motion can be written as

$$x'' - \gamma(x')^2 + g = 0, \tag{1}$$

where primes indicate time derivatives and

$$\gamma = \frac{\hat{\gamma}}{m}\,.$$

[Is (1) also correct when the body is **rising**?]

Since (1) is not linear, it is hard to solve exactly. Let us consider the possibility that a useful approximate solution can be found in the form of the first two terms of a Taylor expansion of x as a function of γ:

$$x(t) = x_0(t) + \gamma x_1(t) + \gamma^2 x_2(t) + \cdots. \tag{2}$$

(We might expect that this will work if γ is sufficiently small. The basic assumption being made here is that the correction terms $\gamma x_1 + \cdots$ are small compared to the "unperturbed" solution x_0.) We substitute (2) into (1) and simplify, neglecting all terms of seond order or higher:

$$(x_0'' + \gamma x_1'' + \cdots) - \gamma(x_0' + \cdots)^2 + g = 0,$$

so after combining terms involving the same power of γ, we get

$$[x_0'' + g] + \gamma[x_1'' - (x_0')^2] + \cdots = 0. \tag{3}$$

Since x_0 and x_1 themselves are not supposed to depend on γ, the only way (3) can hold is that each bracketed term separately equals 0:

$$x_0'' + g = 0, \tag{4}$$

$$x_1'' - (x_0')^2 = 0. \tag{5}$$

Equation (4) is the familiar equation for a falling body without drag. Its solution is

$$x_0(t) = h_0 + v_0 t - \tfrac{1}{2} g t^2, \tag{6}$$

where h_0 and v_0 are the initial height and velocity of the body. Recall that $h_0 \geq 0$ and $v_0 \leq 0$, because of our assumption that the body is **falling** toward ground level.

Substituting (6) into (5), we get

$$x_1'' = (v_0 - gt)^2 = {v_0}^2 - 2gv_0 t + g^2 t^2.$$

We can solve this equation by integrating twice:

$$x_1(t) = h_1 + v_1 t + \tfrac{1}{2}{v_0}^2 t^2 - \tfrac{1}{3} g v_0 t^3 + \tfrac{1}{12} g^2 t^4, \tag{7}$$

where h_1 and v_1 are the constants of integration.

Substituting (6) and (7) into (2), we obtain our first-order approximate solution,

$$x(t) \approx (h_0 + \gamma h_1) + (v_0 + \gamma v_1)t - \tfrac{1}{2}(g - \gamma {v_0}^2)t^2 - \tfrac{1}{3}\gamma g v_0 t^3 + \tfrac{1}{12}\gamma g^2 t^4. \tag{8}$$

This formula becomes more transparent if we consider the special case where the body is simply dropped (initial velocity 0) from height h_0. That is, v_0, h_1, and v_1 are all 0:

$$x(t) \approx h_0 - \tfrac{1}{2} g t^2 + \tfrac{1}{12}\gamma g^2 t^4. \tag{9}$$

The last term in (9) is the modification to the motion caused by drag; in a given time the body does not fall as far as it would in the absence of air resistance.

Notice that this term increases as t^4, so when t is big enough, it will overwhelm the first two terms. This means that, no matter how small γ is, our initial assumption that the drag effect is small will be **wrong** for large times. **There are ways to get around this problem**, but they are too sophisticated to be treated in a freshman course. Our power-series solution should be accurate provided that both γ and t are sufficiently small.

Now let us pose the problem of how fast the body must be **thrown** down so that it will hit the ground at exactly the same time that it would have landed if it were **dropped** in the absence of drag. (This is as close as we can come in a one-dimensional problem to a "targeting algorithm".) Let T be the time of landing in the unperturbed problem:

$$0 = x_0(T) = h_0 - \tfrac{1}{2}gT^2.$$

(Thus $T = \sqrt{2h_0/g}$, but we shall not need to use this formula.) The generalization of (9) to include a negative value of v_1 is

$$x(t) \approx h_0 + \gamma v_1 t - \tfrac{1}{2}gt^2 + \tfrac{1}{12}\gamma g^2 t^4. \tag{10}$$

Thus

$$0 = x(T) \approx \gamma v_1 T + \tfrac{1}{12}\gamma g^2 T^4,$$

or

$$v_1 \approx -\tfrac{1}{2}g^2 T^3. \tag{11}$$

The additional downward initial velocity needed is γv_1 (in this first-order approximation).

One final remark: We have not **proved** that (10) actually is a good approximation to the exact solution. We can't use Taylor's remainder formula, because we have no obvious way of calculating the factor

$$\max_{c\in[0,\gamma]} \left| \frac{d^2x}{d\gamma^2}(c) \right|$$

in it. We can see that the approximation is bad if either γ or t is too big; if γ and t are both fairly small, it seems **highly plausible** that the approximation is good — worth the trouble of testing experimentally or by numerical calculations for a few values of the parameters. Practical engineering can't wait for every theoretical question to be totally settled with mathematical certitude.