Consider the area function, $A(r) = \pi r^2$. When the radius changes from r to $r + dr$, the area changes from πr^2 to $\pi(r + \Delta r)^2$. That is,

$$\begin{aligned}
\Delta A &= \pi(r + \Delta r)^2 - \pi r^2 \\
&= \pi[r^2 + 2r\,\Delta r + (\Delta r)^2] - \pi r^2 \\
&= 2\pi r\,\Delta r + \pi\,(\Delta r)^2,
\end{aligned}$$

so that

$$\frac{\Delta A}{\Delta r} = \pi(2r + \Delta r).$$

Therefore, in the limit that Δr approaches 0, one has

$$\begin{aligned}
\frac{dA}{dr} &= \lim_{\Delta r \to 0} \pi(2r + \Delta r) \\
&= 2\pi r.
\end{aligned}$$