$\TeX$ document #7

The density is now

$$\frac{\text{mass of matter in a square}}{\text{area of the square}}$$

in units $\mathrm{kg/m^2}$ (not $\mathrm{kg/m^3}$). (We think of the integration over the thin, uniform third dimension as having already been done.)

Set up the problem so that the body lies in the x–y plane. D is now the two-dimensional region in that plane occupied by the matter. Then the formula for total mass is

$$M = \iint_D \rho(x,y)\,dA,$$

the coordinates of the center of mass are

$$\overline{x} = \frac{\iint_D \rho(x,y)\,x\,dA}{M}\,, \qquad \overline{y} = \frac{\iint_D \rho(x,y)\,y\,dA}{M}\,,$$

and (since z is always 0 inside the body) the three moments of inertia are

$$I_z = \iint_D \rho\,(x^2+y^2)\,dA, \qquad I_y = \iint_D \rho\,x^2\,dA, \qquad I_x = \iint_D \rho\,y^2\,dA.$$

Notice that if you know any two of the moments of inertia, then you know the third without further calculation! [Insert graphics of lamina spinning about axis perpendicular to lamina and about axis lying in lamina.]