

This example is Exercise 9 of Stewart, Sec. 8.4. However, we will not do it by the **method** recommended in that section. In traditional calculus courses, two-dimensional centroid problems are solved by somewhat awkward, one-dimensional reasoning, so that they can be treated before double and triple integrals. In this course we will take advantage of our earlier exposure to multiple integrals to approach such problems in a more natural way. (But at several places you'll see the Riemann sum from the one-dimensional argument inserted into the calculation inside large parentheses. You're free to ignore these lines.)

Problem: Find the centroid of the region D bounded by

$$y = \cos 2x, \quad y = 0, \quad \text{and} \quad x = \pm \frac{\pi}{4}.$$

[Insert graphic of the region along with a narrow strip at an arbitrary value of x .]

Recall that the centroid is the same as the center of mass if the mass density is constant. Let us use our notation for center-of-mass problems, but take $\rho = 1 \text{ kg/m}^2$. Then M is the same thing as the area of the region, which is

$$M = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x \, dx = \frac{1}{2} \sin 2x \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{2}(1 + 1) = 1.$$

Here we have set up the area by the Riemann-sum reasoning from last semester (see the drawing above). Before proceeding with the numerators of the centroid calculation, however, it is instructive to look at this area from a more fundamental point of view:

[Insert graphic of the region divided into small squares.]

The area is the sum of the areas of its small rectangular pieces, which becomes

$$M = \iint_D \rho \, dA \quad (\text{with } \rho = 1)$$

in the limit of infinitesimal pieces. This is our first nonrectangular double integral! How can we evaluate it? Well, the total area of all the pieces in one column is simply the area of the strip that they form, and then M is

the sum of the areas of those strips. So

$$\begin{aligned}
 M &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\text{height of strip at } x) dx \\
 &= \sum (\text{area of strip at } x) \\
 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\int_0^{\cos 2x} dy \right] dx.
 \end{aligned}$$

Of course, this is the same integral that we just evaluated. The point of looking at it at this more microscopic level is to show that **a double integral can be evaluated as a repeated integral in which the limits on the inner integration depend upon the outer integration variable.**

We can now apply this principle to set up the integral for the numerator of \bar{x} .

$$\begin{aligned}
 M \bar{x} &= \iint_D \rho x dA \\
 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\int_0^{\cos 2x} x dy \right] dx \\
 &= \sum x (\text{area of strip at } x) \\
 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x \cos 2x dx.
 \end{aligned}$$

The hard-but-honest way to evaluate this integral is by parts (Class 20.1 or thereabouts). The easy way is to see that \bar{x} equals 0 by symmetry. In fact, you can see this from the picture, without even writing an integral! Algebraically, it comes about because $x \cos 2x$ is an **odd** function, so the contribution to the integral from each positive value of x is exactly canceled by that from the corresponding negative value. [Insert graphic of the region with its centroid marked on the vertical symmetry axis.]

Finally, we need to find \bar{y} . Again we shall do this by slicing up the

region vertically.

$$\begin{aligned}
 M \bar{y} &= \iint_D y \, dA \\
 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\int_0^{\cos 2x} y \, dy \right] dx \\
 &\left(= \sum \left[\iint_{\text{strip at } x} y \, dA \right] \right) \\
 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} y^2 \Big|_0^{\cos 2x} dx \\
 &= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2x) \, dx.
 \end{aligned}$$

This integral has to be evaluated by a trick presented in Stewart, Section 7.2 (Class 20.2):

$$\begin{aligned}
 M \bar{y} &= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1 + \cos(4x)}{2} dx \\
 &= \frac{1}{4} \left[\frac{\pi}{2} + \int_{-\pi}^{\pi} \cos u \frac{du}{4} \right] \\
 &= \frac{\pi}{8}.
 \end{aligned}$$

Since $M = 1$ in this example, the center of mass is the vector

$$\vec{\mathbf{r}} = \left\langle 0, \frac{\pi}{8} \right\rangle = \frac{\pi}{8} \hat{\mathbf{j}}.$$