Solving a Cubic Equation by Perturbation Theory

You probably do not know how to solve the equation

$$x^3 + \tfrac{1}{10}\,x + 8 = 0$$

exactly. Except in those few cases where one root is obvious (and hence the cubic can be reduced to a quadratic), cubic equations are nearly always solved in practice by a numerical or approximate method.

In this case, the coefficient $\frac{1}{10}$ is smaller than the others. This suggests that we study the equation

$$x^3 + \epsilon x + 8 = 0,$$

find an approximation to the solutions that's accurate when ϵ is small, and set ϵ equal to $\frac{1}{10}$ at the end. Let's assume that

$$x \approx x_0 + \epsilon x_1$$

and find the numbers x_0 and x_1. The idea is that if ϵ is small, then ϵ^2 is even smaller, and terms in the Taylor series involving ϵ^2 or higher powers can probably be ignored.

We calculate

$$x^3 \approx {x_0}^3 + 3\epsilon {x_0}^2 x_1 + 3\epsilon^2 x_0 {x_1}^2 + \epsilon^3 {x_1}^3.$$

Only the first two terms of this formula are "significant", because a term $3\epsilon^2 {x_0}^2 x_2$ has been neglected already in our approximation. So we will throw away all terms that involve power of ϵ higher than the first.

Now the equation becomes

$$\begin{aligned} 0 &= x^3 + \epsilon x + 8 \\ &\approx \quad {x_0}^3 + 3\epsilon {x_0}^2 x_1 \\ &\qquad\quad + \quad \epsilon x_0 \\ &\quad + 8. \end{aligned}$$

The general strategy in perturbative calculations is to make the coefficient of each power of ϵ separately equal to 0, so that the equation is satisfied for all values of ϵ.

The lowest-order equation is

$$0 = {x_0}^3 + 8. \qquad (\epsilon^0)$$

Its principal solution is $x_0 = -2$. (The equation also has two complex roots, but we will ignore them today.)

Substitute this result into the next equation:

$$0 = 3{x_0}^2 x_1 + x_0 = 12x_1 - 2. \qquad (\epsilon^1)$$

Thus $x_1 = \frac{1}{6}$.

So we have found a **first-order perturbative solution**,

$$x \approx -2 + \frac{\epsilon}{6},$$

which is actually the Taylor polynomial $T_1(\epsilon)$ of the exact solution.

Let us check this solution by substituting it into the original cubic equation. After working out the algebra we get

$$x^3 + \epsilon x + 8 = \frac{\epsilon^3}{216}.$$

The right-hand side (called the **residual**) is not exactly zero, but it is small compared to ϵ if ϵ itself is small.*

If $\epsilon = \frac{1}{10}$, our approximation is $x \approx -2.01666\ldots$. Compare this with the "exact" answer calculated by Maple.

* We had no right to expect the residual to be smaller than the order ϵ^2, but by accident it is of order ϵ^3. If you go back and put terms $\epsilon^2 x_2 + \epsilon^3 x_3$ into the assumed form of the answer, you will find that $x_2 = 0$ but x_3 is not zero.