The Damped Harmonic Oscillator

Consider the differential equation

$$\frac{d^2y}{dt^2} + 2\epsilon\,\frac{dy}{dt} + y = 0.$$

For definiteness, consider the initial conditions

$$y(0) = 0, \qquad y'(0) = 1.$$

Try

$$y = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots.$$

We get (to second order)

$$\begin{aligned} 0 &\approx (y_0'' + \epsilon y_1'' + \epsilon^2 y_2'') + (2\epsilon y_0' + 2\epsilon^2 y_1') + (y_0 + \epsilon y_1 + \epsilon^2 y_2) \\ &= (y_0'' + y_0) + \epsilon(y_1'' + 2y_0' + y_1) + \epsilon^2(y_2'' + 2y_1' + y_2). \end{aligned}$$

The initial conditions don't depend on ϵ, so they break into $y_j(0) = 0$ and

$$y_0'(0) = 1, \qquad y_1'(0) = 0, \qquad y_2'(0) = 0.$$

Now we set the coefficient of each power of ϵ equal to 0 and apply the corresponding boundary condition.

The solution for y_0 is

$$y_0(t) = \sin t.$$

Substitute this into the equation for y_1 :

$$y_1'' + y_1 = -2y_0' = -2\cos t.$$

Now remember the **method of undetermined coefficients** for an inhomogeneous linear equation with the forcing term "on resonance":

$$y_1 = A\,t\,\cos t + B\,t\,\sin t + \text{homogeneous solution}.$$

After algebra, you find that the solution satisfying the null initial conditions is

$$y_1(t) = -t\,\sin t.$$

(Check it.) This is called a **secular term**, because it grows with t.

For the second-order term we get the equation

$$y_2'' + y_2 = -2y_1' = 2\sin t + 2\,t\,\cos t.$$

We know that the solution will involve t^2 times a trig function. And so on to higher orders. (The secular terms are getting worse!) So we have constructed

$$y(t;\epsilon) = \sin t - \epsilon t\,\sin t + \text{term involving } \epsilon^2 t^2 + \cdots.$$

To judge this approximation, let's look at the exact solution. It is

$$y(t;\epsilon) = \frac{e^{-\epsilon t}}{\sqrt{1-\epsilon^2}}\,\sin(\sqrt{1-\epsilon^2}\,t).$$

Expanding this in a Taylor series in ϵ (with t fixed), we get agreement with our perturbative solution, as far as we've carried it. (Work it out.) For any given t, our approximation is good if ϵ is sufficiently small. But for a fixed ϵ, there eventually comes a t for which the error is large. Our method has led us to expand $e^{-\epsilon t}$ as a power series, but that is **a bad thing to do**, clearly. More advanced techniques of perturbation theory are needed to get around this problem.