Discovering Taylor's Theorem

Consider a function $x = f(t)$. Suppose that

$$f(0) = x_0\,, \qquad f'(0) = v_0\,, \qquad f''(0) = a_0\,.$$

We expect that somehow

$$f(t) \approx x_0 + v_0 t + \tfrac{1}{2} a_0 t^2, \tag{1}$$

at least for t very close to 0. There are related equations for the derivatives:

$$f'(t) \approx v_0 + a_0 t, \tag{2}$$

$$f''(t) \approx a_0\,. \tag{3}$$

(**Remark:** (1) is quite different from the **false** equation

$$f(t) = x_0 + v_0 t + \tfrac{1}{2} a(t) t^2,$$

frequently written down by students who don't understand that the falling-body formula applies only when the acceleration is **constant**.)

We'll start with the simplest formula, (3), and work back up. Since f'' starts at 0 when $t = 0$, how far away can it wander by the time $t = 2$, say? Clearly that depends on the function; to get a useful conclusion, we need to impose another assumption. We shall assume that the **third** derivative of f exists and

$$|f'''(t)| \le M \quad \text{for all } t \tag{4}$$

for some constant M. Then, for $t > 0$,

$$\int_0^t f'''(u)\,du \le \int_0^t M\,du \tag{5}$$

(see "Order properties of the integral," p. 279 of Stewart ed. 3). That is,

$$f''(t) - a_0 \le Mt. \tag{6}$$

So, for instance, $f''(2)$ can't be bigger than $a_0 + 2M$.

We can now integrate (6) in the same way:

$$\int_0^t [f''(u) - a_0]\,du \le \int_0^t Mu\,du; \tag{7}$$

$$f'(t) - (v_0 + a_0 t) \le \tfrac{1}{2}Mt^2. \tag{8}$$

(Thus $f'(2)$ can't be bigger than $v_0 + 2a_0 + 2M$, which goes a long way to toward clarifying and justifying (2).) Finally, integrate (8):

$$\int_0^t [f'(u) - (v_0 + a_0 u)]\,du \le \int_0^t \tfrac{1}{2}Mu^2\,du; \tag{9}$$

$$f(t) - (x_0 + v_0 t + \tfrac{1}{2}a_0 t^2) \le \tfrac{1}{6}Mt^3. \tag{10}$$

So far we have concentrated on getting **upper** bounds on the differences between the exact functions and their polynomial approximations, for $t > 0$. This restriction was made just to keep the formulas simple-looking. To handle both lower bounds and negative values of t requires going back through the argument and doing careful bookkeeping with minus signs and absolute values. We will skip that. The conclusions then are

$$|f(t) - (x_0 + v_0 t + \tfrac{1}{2}a_0 t^2)| \le \tfrac{1}{6}M|t|^3, \tag{11}$$

$$|f'(t) - (v_0 + a_0 t)| \le \tfrac{1}{2}Mt^2, \tag{12}$$

$$|f''(t) - a_0| \le M|t|, \tag{13}$$

Equations (11)–(13) give a precise meaning and justification to claims (1)–(3). Recall that M is any number satisfying (4); that is, an upper bound on the third derivative of f.

Pictorially, (13) says that the graph of f'' is trapped between the lines with slopes $\pm M$ meeting at the point $(0, a_0)$. Similarly, (11) says that the graph of $f(t) - (x_0 + v_0 t + \frac{1}{2}a_0 t^2)$ fits between the two cubic curves $\pm\frac{1}{6}M|t|^3$ (which, of course, come very close together near $t = 0$).

[I will insert appropriate graphics someday.]