Problem: Approximate the integral $\int_0^1 e^{x^2}\,dx$ to 2 significant figures.

Method: We will approximate the integrand by a Taylor polynomial — so that we can integrate it! — and use Taylor's theorem with remainder to justify the answer.

Substituting x^2 for z in the Taylor expansion of e^z, we get

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + R_3(x^2).$$

We can write out as many terms as we need; if this turns out to be not enough, we'll come back for more later. Therefore,

$$\begin{aligned}\int_0^1 e^{x^2}\,dx &= \int_0^1 \left[1 + x^2 + \cdots\right] dx \\ &= \left[x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42}\right]_0^1 + \int_0^1 R_3(x^2)\,dx. \\ &= 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} + \int_0^1 e^c\,\frac{x^8}{24}\,dx.\end{aligned}$$

At the last step we have used the standard formula for the Taylor remainder ("Version 2" in the terminology of this Web page), and the fact that $f^{(4)}(z) = e^z$ if $f(z) = e^z$. The number c may depend on x, but in this problem it is always guaranteed to be between 0 and 1; therefore, e^c is less than e, and the remainder term in the integral is less than

$$e\int_0^1 \frac{x^8}{24}\,dx = \frac{e}{216}\,.$$

So, doing the arithmetic with the fractions, we conclude that

$$\int_0^1 e^{x^2}\,dx \approx 1.45714 \approx 1.46$$

with a maximum error of

$$\frac{e}{216} = 0.01258.$$

So the accuracy is right on the edge of what we wanted. To be safe, one should go back and include one more term in the appoximation:

$$\int_0^1 e^{x^2}\,dx \approx \cdots + \frac{1^9}{216} \approx 1.46$$

with a maximum error of

$$\int_0^1 R_4(x^2)\,dx \leq \frac{e}{11 \times 5!} \approx 0.002.$$

You might compare this result with the value for the integral calculated numerically by Maple.