

Cones, Wedges, and Torques

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I. Stress tensor calculations

REFERENCES

1. SAF, CST, Mai Truong, and Jef Wagner,
“Wedges, Cones, Cosmic Strings, and the Reality
of Vacuum Energy”,
Dowker festschrift paper, arXiv 1205:1818
2. CST, “Vacuum Energy for Static, Cylindrically
Symmetric Systems”,
Undergraduate Fellow Thesis, <http://www.math.tamu.edu/~fulling/trendathesis.pdf>

SOURCES

1. A. Sommerfeld, 1896–1901
2. H. S. Carslaw, 1899–1920
3. W. Lukosz, *Z. Phys.* **262** (1973) 327
4. J. S. Dowker, 1977–1987
5. A. G. Smith, in *The Formation and Evolution of Cosmic Strings*, Cambridge, 1990

Our main contributions:

- local energy density and pressure
- dependence on direction of cutoff

CYLINDER KERNELS \overline{T} IN CYLINDRICAL COORDINATES

Full Euclidean space

$$\begin{aligned}\overline{T}_{2\pi} &= -\frac{1}{2\pi^2(t^2 + |\mathbf{x} - \mathbf{x}'|^2)} \\ &= -\frac{1}{2\pi^2(r^2 + r'^2 - 2rr' \cos(\theta - \theta') + t^2 + (z - z')^2)}.\end{aligned}$$

(We can take $z' = 0 = \theta'$.)

Direct solution in cylindrical coordinates gives

$$\begin{aligned}\bar{T}_{2\pi} &= -\frac{1}{4\pi^2 rr' \sinh u} \sum_{n=-\infty}^{\infty} e^{-|n|u+in\theta} \\ &= -\frac{1}{4\pi^2 rr' \sinh u} \frac{\sinh u}{\cosh u - \cos \theta}\end{aligned}$$

where u is defined by any of

$$u = -\ln \frac{r_2 - r_1}{r_2 + r_1} \quad \text{with}$$

$$r_1 = \sqrt{(r - r')^2 + z^2 + t^2},$$

$$r_2 = \sqrt{(r + r')^2 + z^2 + t^2};$$

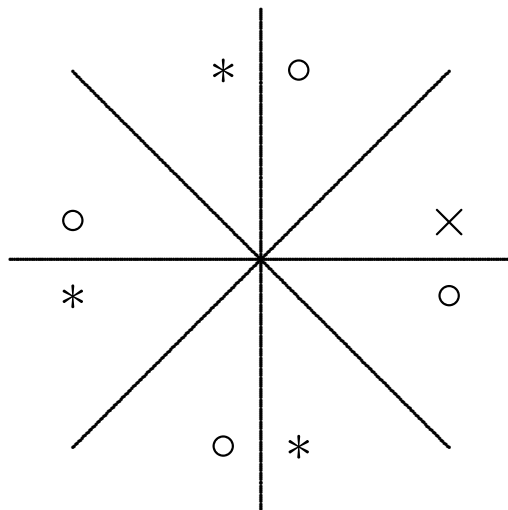
$$2rr' \cosh u = r^2 + r'^2 + z^2 + t^2;$$

$$2rr' \sinh u = \sqrt{[r^2 + r'^2 + z^2 + t^2]^2 - 4r^2 r'^2};$$

$$4rr' \sinh^2 \left(\frac{u}{2} \right) = (r - r')^2 + z^2 + t^2.$$

Dirichlet wedge with a “good” angle, $\alpha = \frac{\pi}{N}$

— can be solved by the method of images:



Dowker space

Replace $0 \leq \theta < 2\pi$ by $-\infty < \theta < \infty$ (Riemann surface of $\ln z$; locally flat with singularity on axis).

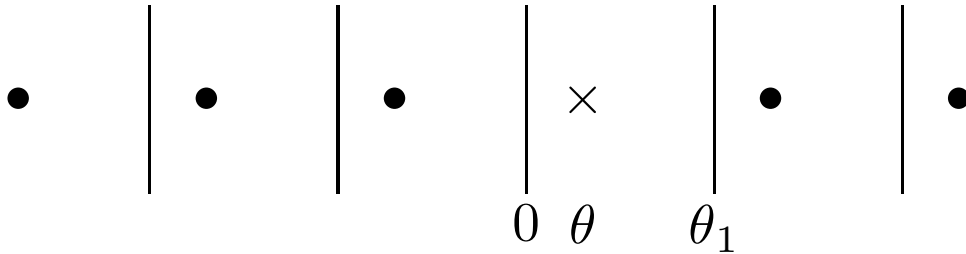
Replace Fourier series by Fourier transform:

$$\begin{aligned}\bar{T}_\infty &= -\frac{1}{4\pi^2 rr' \sinh u} \int_{-\infty}^{\infty} e^{-|\lambda|u+i\lambda\theta} d\lambda \\ &= -\frac{1}{2\pi^2 rr' \sinh u} \frac{u}{u^2 + \theta^2}.\end{aligned}$$

Cone of angle θ_1

$0 \leq \theta < \theta_1$; singular on the axis unless $\theta_1 = 2\pi$ (visualizable if $\theta_1 < 2\pi$).

1. Solve again by Fourier series, **or**
2. Obtain from Dowker space by periodic image sum:



$$\bar{T}_{\theta_1} = - \frac{1}{2\pi\theta_1 rr' \sinh u} \frac{\sinh\left(\frac{2\pi u}{\theta_1}\right)}{\cosh\left(\frac{2\pi u}{\theta_1}\right) - \cos\left(\frac{2\pi\theta}{\theta_1}\right)}$$

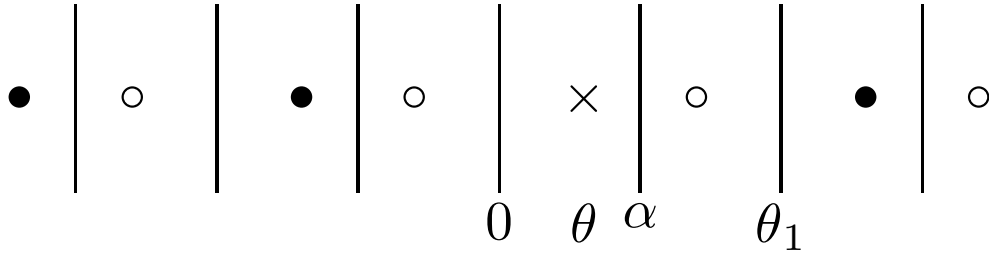
where u is exactly the same as before.

$\theta_1 = 2\pi$ recovers Euclidean space.

$\theta_1 \rightarrow \infty$ recovers Dowker space.

Dirichlet wedge of arbitrary angle α

— from cone of angle $\theta_1 = 2\alpha$ (θ' restored) by images:



$$\bar{T}_\alpha = -\frac{1}{4\pi\alpha rr' \sinh u} \left(\frac{\sinh(\pi u/\alpha)}{\cosh(\pi u/\alpha) - \cos(\pi(\theta - \theta')/\alpha)} - \frac{\sinh(\pi u/\alpha)}{\cosh(\pi u/\alpha) - \cos(\pi(\theta + \theta')/\alpha)} \right)$$

ENERGY DENSITY AND PRESSURE

$$\langle T_{00} \rangle = -\frac{1}{2} \partial_t^2 \bar{T} + \beta [\partial_r \partial_{r'} \bar{T} + \partial_r^2 \bar{T} + \frac{1}{r} \partial_r \bar{T}],$$

$$\langle T_{rr} \rangle = -\frac{1}{4} [\partial_r \partial_{r'} \bar{T} - \partial_r^2 \bar{T}] - \frac{1}{r} \beta \partial_r \bar{T},$$

$$\langle T_{\perp\perp} \rangle = \frac{1}{4r} \partial_r \bar{T} + \frac{1}{4r^2} [\partial_\theta^2 \bar{T} - \partial_\theta \partial_{\theta'} \bar{T}] - \beta [\partial_r \partial_{r'} \bar{T} + \partial_r \bar{T}],$$

$$\langle T_{zz} \rangle = -\frac{1}{4} [\partial_z \partial_{z'} \bar{T} - \partial_z^2 \bar{T}] - \beta [\partial_r \partial_{r'} \bar{T} + \partial_r^2 \bar{T} + \frac{1}{r} \partial_r \bar{T}].$$

We ignore off-diagonal components for now.

Subtract $\overline{T}_{2\pi}$ terms (zero-point energy).

Take $r' = r$, $\theta' = \theta$, $z' = 0$; cutoff $\sqrt{t^2 + z^2}$ remains.

Ultraviolet cutoff: $t \neq 0$, $z = 0$.

Neutral cutoff (except for T_{zz}): $z \neq 0$, $t = 0$.

Stress tensor in Dowker space, $\beta = 0$

t -cutoff: Festschrift p. 16

z -cutoff: Thesis p. 22 (30) (T_{rr} and $T_{\perp\perp}$ unchanged)

Dashed curves are for zero cutoff.

Close examination shows that as $r \rightarrow 0$,
 $\rho \equiv T_{00}$ and $p_z \equiv T_{zz}$ approach finite limits (cusps),
but $p_r \equiv T_{rr}$ and $p_\theta \equiv T_{\perp\perp}$ diverge. Why?

Recall $\bar{T}_\infty = -\frac{1}{2\pi^2 r r' \sinh u} \frac{u}{u^2 + (\theta - \theta')^2}$,

where $u \equiv \cosh^{-1} a$, $a = 1 + \frac{z^2 + t^2}{2r^2}$ when $r' = r$, $\theta' = \theta$.

As $r \rightarrow 0$, $a \rightarrow \infty$ and $u \sinh u \sim a \ln a$.

Stress components can behave like $r^{-2} \ln r$ at worst
(hence $r dr d\theta$ integrals can diverge).

The point-splitting cutoff has not completely removed
the divergence!

Dowker space β term

t-cutoff: Festschrift p. 17

z-cutoff: No change

All components diverge at the axis.

Stress tensor in cones, $\beta = 0$

t-cutoff: Festschrift pp. 18–19

z-cutoff: Thesis p. 22 (30)

All components remain finite at the axis.

(Same for β terms.)

Stress tensor in wedges

Now things depend on θ .

t -cutoff: Festschrift pp. 22–25

z -cutoff: Thesis p. 23–24 (31–32) (right angle, $\beta = 0$)

The cutoff quantities are finite at the plates and at the axis.

II. Torque

In a wedge, the parameter we can vary is the angle. One expects

$$-\frac{dE}{d\alpha} = \text{torque/length} = \int_0^\infty T_{\perp\perp} r dr \equiv \tau.$$

With luck, the integrands will be equal pointwise.

$$E = \text{total energy/length} = \int_0^\infty r dr \int_0^\alpha d\theta T_{00}.$$

Henceforth I omit “/length”. (V will be an area!)

CLASSICAL PERFECT FLUID

Assume equation of state $p = \beta\rho$.

p is isotropic, ρ and p homogeneous (but depend on α).

Consider wedge with large radius R .

$$E = \rho V = \frac{1}{2}R^2\alpha\rho \Rightarrow \frac{dE}{d\alpha} = \frac{1}{2}R^2 \left(\rho + \alpha \frac{d\rho}{d\alpha} \right).$$

$$\tau = \int_0^R p r dr = \frac{1}{2}R^2\beta\rho.$$

So we expect $r + \alpha \frac{d\rho}{d\alpha} = -\beta\rho$.

$$\frac{1}{\rho} \frac{d\rho}{d\alpha} = -\frac{\beta + 1}{\alpha} \Rightarrow \rho = K\alpha^{-(\beta+1)}.$$

$$E = \frac{1}{2}R^2K\alpha^{-\beta} \Rightarrow E = CV^{-\beta} \text{ or } \rho = CV^{-(\beta+1)}.$$

These two final formulas are shape-independent and are *equivalent* to the equation of state.

We could generalize to ρ, p, β dependent on r (e.g., surface tension).

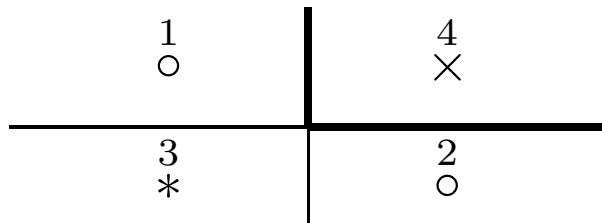
TORQUE IN A WEDGE

We expect $\tau = -\frac{dE}{d\alpha}$ for point-splitting in the neutral (z) direction.

But *Mathematica* says Nay!

The trouble is probably not in the quantum field theory, but in a misinterpretation of the classical physics of stress and torque.

To understand, we consider $\alpha = \frac{\pi}{2}$ (checks with Cartesian answer!) and look at each term in \bar{T} separately. Physical intuition is fairly clear in this case.



When $\alpha = \frac{\pi}{2}$, term 4 is canceled by zero-point energy but its α derivative is nonzero. (The background is the cone of angle 4α .)

Terms 1 and 2 in \overline{T} depend on $\theta + \theta'$ and have ρ concentrated near a plate, p parallel to that plate.

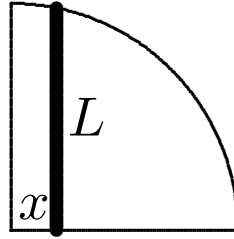
“Pure cone” terms 3 and 4 depend on $\theta - \theta'$ and have ρ, p independent of θ .

So far we have ignored off-diagonal components of $T_{\mu\nu}$. They are not always zero!

Term 1 has $p_x = 0$, $p_y = -\rho$, $T_{xy} = 0$. Therefore, $T_{r\perp} \neq 0$ away from the plates (a nonunit diagonal matrix becomes nondiagonal under similarity transformation). Similarly for term 2.

Analogously, one would expect $T_{xy} \neq 0$ away from the plates for terms 3 and 4. But it turns out that $p_x = p_y$ for these terms (in fact, they are 0 after zero-point subtraction).

ANOTHER CLASSICAL ANALOGUE



Consider a quarter-circle apparatus involving a band of material with a uniform energy density per unit length. $L =$ length; $x =$ distance from left boundary. As before,

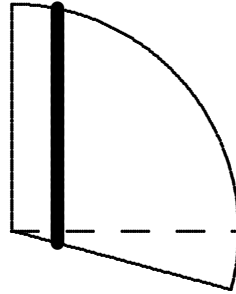
$$E = CL^{-\beta} \iff p = C\beta L^{-\beta-1} = \beta\rho.$$

Instances:

1. Band is a physical object, like a rubber band or spring ($\beta = -2$ is Hooke's law.) (But keep $p_x = 0$.)
2. The sides and arc are “conductors” and the band is an infinitesimal part of the vacuum energy generated by the Dirichlet boundary along the left side. Here $\beta = -1$ (surface tension, 1D dark energy).

The point is that even the first case is problematical.

Move bottom side by angle $\Delta\alpha$:

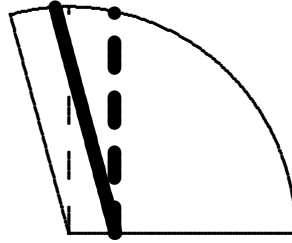


Torque on bottom boundary is $\tau = xp$. Rate of change of total energy is, as it should be,

$$\frac{dE}{d\alpha} = \frac{dE}{dL} \frac{dL}{d\alpha} = -px = -\tau.$$

In scenario 2, the band must stay at a constant perpendicular distance from the left boundary. In scenario 1, if the spring is rigidly attached, the attachment point should remain at a constant distance from the vertex. However, when $\Delta\alpha$ is infinitesimal the differences between these situations are of second order.

Move left side by angle $\Delta\alpha$:



$dE/d\alpha$ is same as before. But in vacuum scenario (2) there is no pressure against the left side of the wedge. It therefore appears that there is no torque and hence is a violation of the expected energy-torque relation.

In spring scenario (1) some physical mechanism must force the band to remain parallel to the moved boundary, and hence there must be a reaction force on that boundary. Because the band is not perpendicular to the arc, there is a shear force acting on the band to turn it, and an opposite force acting on the boundary mechanism as a whole.

Probably the vacuum counterpart is this: The component $T_{r\theta} \neq 0$ represents a shear force.

Details of the classical case:

The equation of the arc is $y = \sqrt{R^2 - x^2}$ and hence its slope is

$$\frac{dy}{dx} = -\frac{x}{\sqrt{R^2 - x^2}} = -\frac{x}{L} \equiv -\tan \gamma \approx -\gamma.$$

Therefore the component of force along the arc is

$$-p \sin \gamma \approx -\frac{px}{L}. \quad (*)$$

To lowest order the top end of the band moves a distance $-L\Delta\alpha$, which with (*) suggests a work $px\Delta\alpha$, hence a torque xp . The energy-torque balance is thus restored.

However, this account tacitly assumes that the arc, or at least the part of it between the band and the left boundary, rotates rigidly. In the vacuum energy case it is not clear exactly what the torque from the shear stress is acting upon. (The boundary energy moves with the flat boundary, regardless of whether the circular boundary is rigid, stretchable, or completely absent.) Furthermore, this analysis says nothing to resolve the torque paradox for terms 3 and 4 (the parts of the wedge stress tensor — due to the vertex, not the wedge sides — that are diagonal in polar coordinates).

The crux of the problem for terms 3 and 4 is that E depends on α in two ways: ρ depends on α (cone angle!), and for a given ρ , the region of integration depends on α . For rectangular boundaries only the second effect came into play. The cancellations that insured energy-pressure balance in the rectangular case now leave the first effect uncompensated.