Mass dependence of instanton determinant in QCD: part II

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- determinants in quantum field theory
- functional determinant to radial problem
- radial WKB methods: renormalization
- radial WKB methods: approximation
- comparing with other methods

with: Jin Hur and Choonkyu Lee (SNU) PLB, hepth/0407222; 0410190

Instanton background in QCD

scalar (Klein-Gordon) determinant in an instanton background :

$$\Gamma^S(A;m) = \ln\left[\frac{\mathrm{Det}(-D^2+m^2)}{\mathrm{Det}(-\partial^2+m^2)}\right]$$

now involves **partial** differential operators

radial symmetry reduces problem to a sum over ODEs

Regularization and renormalization

$$\Gamma_\Lambda = \sum_{l=0,\frac{1}{2},\dots} (2l+1)(2l+2)\left\{\ln\det\left(\frac{\mathcal{H}_{(l,l+\frac{1}{2})}+m^2}{\mathcal{H}^{\text{free}}_{(l)}+m^2}\right) + \ln\det\left(\frac{\mathcal{H}_{(l+\frac{1}{2},l)}+m^2}{\mathcal{H}^{\text{free}}_{(l+\frac{1}{2})}+m^2}\right)\right.$$

$$\left. - \ln\det\left(\frac{\mathcal{H}_{(l,l+\frac{1}{2})}+\Lambda^2}{\mathcal{H}^{\text{free}}_{(l)}+\Lambda^2}\right) - \ln\det\left(\frac{\mathcal{H}_{(l+\frac{1}{2},l)}+\Lambda^2}{\mathcal{H}^{\text{free}}_{(l+\frac{1}{2})}+\Lambda^2}\right)\right\}$$

problem : large l and large Λ limits ?

solution : split sum into 2 parts, with L large but finite

$$\Gamma^S_\Lambda(A;m) = \sum_{l=0,\frac{1}{2},\dots}^{L} \Gamma^S_{(l)}(A;m) + \sum_{l=L+\frac{1}{2}}^{\infty} \Gamma^S_{\Lambda,(l)}(A;m)$$

evaluate **numerically**, for large L

evaluate **analytically**, for large L

WKB computation

- Separation of lower angular momentum
  + higher angular momentum

- using "Radial" WKB method, we compute **analytically and exactly** the higher angular momentum piece

- alternatively, WKB provides an **approximation** method for the lower angular momentum sector

Phase shifts and radial WKB

regularized effective action :

$$\Gamma_{\Lambda}^{S}(A;m) = -\int_0^\infty \frac{ds}{s}\left(e^{-m^2 s} - e^{-\Lambda^2 s}\right) \overbrace{\int d^4x \, \mathrm{tr}\langle x | e^{-s(-\mathrm{D}^2)} - e^{-s(-\partial^2)} | x\rangle}^{F(s)}$$

express "the heat kernel" F(s) in terms of **radial phase shifts**

$$F(s) = \frac{2s}{\pi} \sum_{l=0,\frac{1}{2},\cdots} (2l+1)(2l+2) \int_0^\infty dk \; e^{-k^2 s} k \left[\eta_{l,l+\frac{1}{2}}(k) + \eta_{l+\frac{1}{2},l}(k)\right]$$

evlaute the phase shifts from the radial Schrodinger equation

$$\left\{-\frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} + \frac{4l(l+1)}{r^2} + \frac{4(j-l)(j+l+1)}{r^2+1} - \frac{3}{(r^2+1)^2}\right\}\psi(r) = k^2\psi(r)$$

Radial WKB method

$$\left\{\frac{d^2}{dx^2} + Q(x)\right\}\Psi(x) = 0,$$

Standard form for WKB

$$\Psi(x) = r\psi(r)\big|_{r=e^x} = e^x\psi(r = e^x),$$

Change of variables

$$Q_{(l,j)}(x) = e^{2x}\left\{k^2 - \frac{4\left(l+\frac{1}{2}\right)^2}{e^{2x}} - \frac{4(j-l)(j+l+1)}{e^{2x}+1} + \frac{3}{(e^{2x}+1)^2}\right\}$$

radial WKB is good for **large values of angular momentum l**

Leading terms in WKB

$$\eta^{(1)}_{l,j} = \frac{1}{2}\oint \sqrt{k^2 - \tilde{V}_{(l,j)}(r)}\; dr - (\text{free})$$

$$\eta^{(2)}_{l,j} = \frac{1}{2}\oint \left\{ \frac{1}{8r^2}\frac{1}{(k^2 - \tilde{V}_{(l,j)}(r))^{1/2}} + \frac{1}{48}\frac{d^2\tilde{V}_{(l,j)}(r)}{dr^2}\frac{1}{(k^2 - \tilde{V}_{(l,j)}(r))^{3/2}} \right\} dr - (\text{free})$$

Langer modified potential

$$\tilde{V}_{(l,j)}(r) \equiv \frac{4\left(l+\frac{1}{2}\right)^2}{r^2} + \frac{4(j-l)(j+l+1)}{r^2+1} - \frac{3}{(r^2+1)^2}$$

k integrals and l-summation

can do k integrals :

$$\frac{1}{2}\oint dk\,\frac{2s}{\pi}\,e^{-k^2 s}\frac{k}{\left[k^2-\tilde{V}(r)\right]^{n+\frac{1}{2}}}=\frac{e^{-s\tilde{V}(r)}s^{n+\frac{1}{2}}\Gamma(-n+\frac{1}{2})}{\pi}$$

WKB approximation for proper-time kernels :

$$F_L(s) = \int_0^\infty dr\left(\sum_{l=L+\frac{1}{2}}^{\infty} f_l(s,r)\right)$$

$$f_l(s,r) = (2l+1)(2l+2)[f_{(l,l+\frac{1}{2})}(s,r)+f_{(l+\frac{1}{2},l)}(s,r)]$$

$$f^{(1)}_{(l,j)}(s,r)=\frac{1}{2\sqrt{\pi s}}\,\exp\left[-s\tilde{V}_{(l,j)}(r)\right]-(\text{free})$$

$$f^{(2)}_{(l,j)}(s,r)=\frac{1}{2\sqrt{\pi s}}\left(\frac{s}{4r^2}-\frac{s^2}{12}\frac{d^2\tilde{V}_{(l,j)}}{dr^2}\right)\exp\left[-s\tilde{V}_{(l,j)}(r)\right]-(\text{free})$$

use WKB for large l part of sum : do l sum using **Euler-MacLaurin**

$$\sum_{l=L+\frac{1}{2}}^{\infty} f_l = 2\int_L^\infty dl\, f(l)-\frac{1}{2}f(L)-\frac{1}{24}f'(L)+\dots$$

remaining r and s integrals done in large L limit

Large L behavior from WKB

analytic WKB (**exact** in large L) computation :

$$\sum_{l=L+\frac{1}{2}}^{\infty} \Gamma^S_{\Lambda,(l)}(A;m) \sim \frac{1}{6}\ln\Lambda + 2L^2 + 4L - \left(\frac{1}{6} + \frac{m^2}{2}\right)\ln L$$

$$+\left[\frac{127}{72} - \frac{1}{3}\ln 2 + \frac{m^2}{2} - m^2\ln 2 + \frac{m^2}{2}\ln m\right] + O\left(\frac{1}{L}\right)$$

2nd order WKB (higher orders don't contribute in large L limit)

NOTE :

- $\ln\Lambda$ term exactly as required for renormalization
- quadratic, linear and log divergences, and finite part
- exactly cancel divergences from numerical sum in large L limit !!!
- note mass dependence in "subtraction" terms

When m=0 , analytic expression

analytic check : **massless** case

$$S_{(l,l+\frac{1}{2})}(r) = \ln\left[\frac{2l+1}{2l+2}\right] + \ln\left[\sqrt{1+r^2} + \frac{1}{2l+1}\frac{1}{\sqrt{1+r^2}}\right]$$

$$S_{(l+\frac{1}{2},l)}(r) = -\ln\left[\sqrt{1+r^2}\right]$$

exact renormalized effective action :

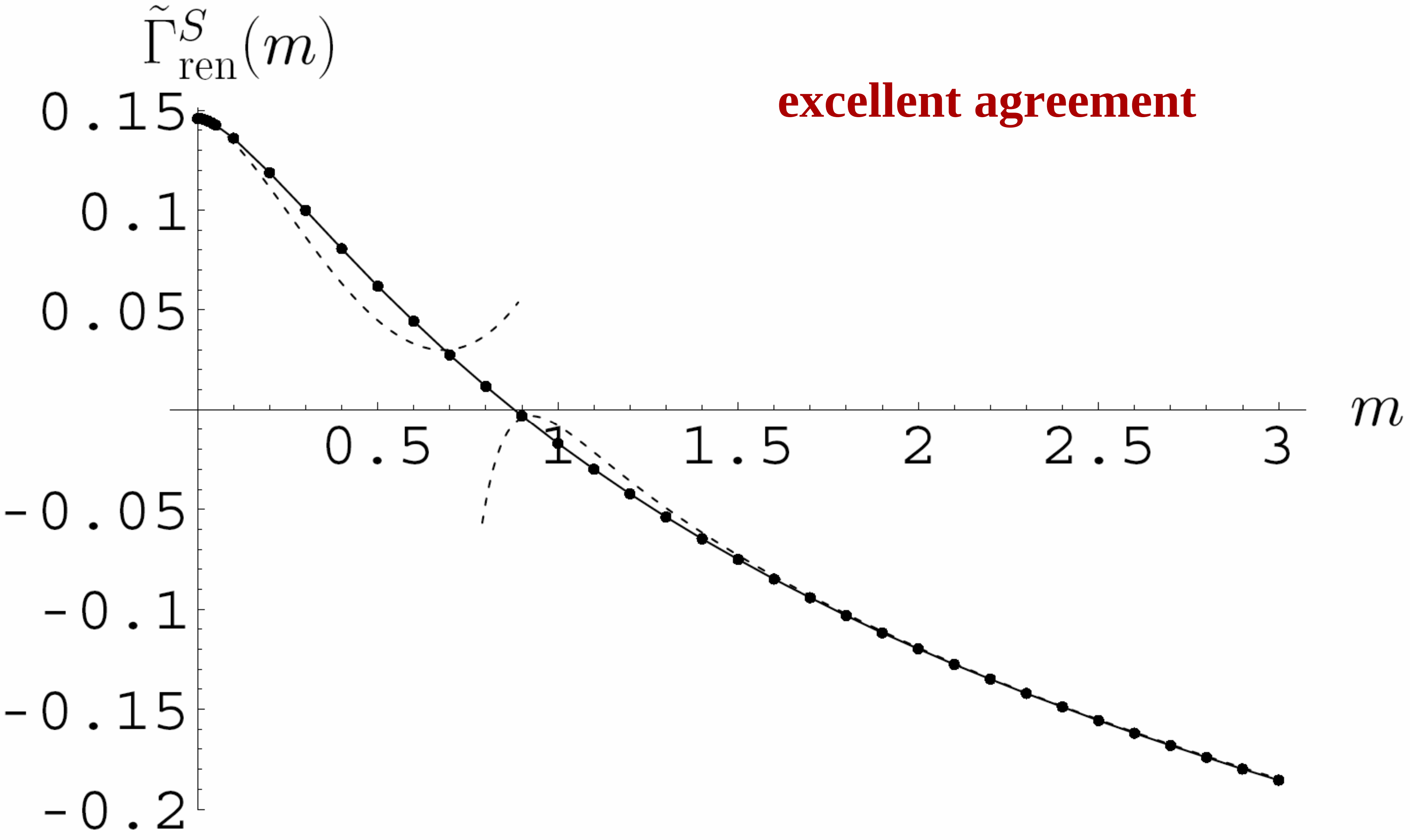
$$\Gamma^S_\Lambda(A;m) = \sum_{l=0,\frac{1}{2},\dots}^{L} \Gamma^S_{(l)}(A;m) + \sum_{l=L+\frac{1}{2}}^{\infty} \Gamma^S_{\Lambda,(l)}(A;m)$$

$$\tilde{\Gamma}^S_{\text{ren}}(m=0) = \lim_{L\to\infty}\left\{\sum_{l=0,\frac{1}{2},\dots}^{L}(2l+1)(2l+2)\ln\left(\frac{2l+1}{2l+2}\right) + 2L^2 + 4L - \frac{1}{6}\ln L + \frac{127}{72} - \frac{1}{3}\ln 2\right\}$$

$$= -\frac{17}{72} - \frac{1}{6}\ln 2 + \frac{1}{6} - 2\zeta'(-1)$$

$$= \alpha\left(\frac{1}{2}\right) = 0.145873\dots$$

Comparison with asymptotic results(massive case)



Singular gauge computation

singular gauge

$$A_\mu^{\rm sing}(x) \equiv A_\mu^a(x)\frac{\tau^a}{2} = \frac{\bar\eta_{\mu\nu a}\tau^a x_\nu}{r^2(r^2+\rho^2)}$$

potential becomes more singular ? **NO**

$$V_{(l,j)}^{\rm singular}(r) = V_{(j,l)}^{\rm regular}(r)$$

(consequence of **conformal invariance**)

$$\Gamma \approx \Sigma_l \left(\ln\det\left(\mathrm{H}_{(l,l+1/2)} + m^2 \right) + \ln\det\left(\mathrm{H}_{(l+1/2,l)} + m^2 \right)\right)$$

Effective action has **same value** in regular and singular gauge

WKB as an approximation

PLB600, 302 / hepth0407222

$$F(s) = \frac{2s}{\pi} \sum_{l=0,\frac{1}{2},\cdots} (2l+1)(2l+2) \int_0^\infty dk\ e^{-k^2 s} k \left[\eta_{l,l+\frac{1}{2}}(k) + \eta_{l+\frac{1}{2},l}(k)\right]$$

include lower angular momentum sector

$$F^{(1)}(s) = \frac{1}{2\sqrt{\pi}\sqrt{s}} \int_0^\infty dr \left(\sum_{l=0,\frac{1}{2},\cdots} (2l+1)(2l+2) \left\{ e^{-sV^{l,l+\frac{1}{2}}(r)} - e^{-sV_0^l(r)} + e^{-sV^{l+\frac{1}{2},l}(r)} - e^{-sV_0^{l+\frac{1}{2}}(r)} \right\} \right)$$

1st order WKB does not give correct value for the renormalization

$$F^{(1)}(s=0) = -\frac{1}{24} \qquad (\neq -\frac{1}{12})$$

$$\Gamma_\Lambda^S(A;m) = -\int_0^\infty \frac{ds}{s}(e^{-m^2 s} - e^{-\Lambda^2 s})F(s) \quad = -F(s=0)\ln\frac{\Lambda^2}{m^2} + \ldots..$$

Higher order WKB

$1^{st} + 2^{nd}$ WKB gives correct value at s=0

$$F^{(1)+(2)}(s=0) = -\frac{1}{12} = F_{\text{Correct}}(s=0)$$

$1^{st} + 2^{nd} + 3^{rd}$ order WKB gives good approximation for small s / large mass

$$\frac{d}{ds}F^{(1)+(2)+(3)}(s=0) = \frac{1}{75}$$

$$s \to 0+ \quad : \quad F(s) \sim -\frac{1}{12} + \frac{1}{75}s + \frac{17}{735}s^2 - \frac{116}{2835}s^3 + \cdots$$ ← **heat kernel**

$$m \to \infty \ : \ \tilde{\Gamma}^S(m) = -\frac{1}{6}\ln m - \frac{1}{75m^2} - \frac{17}{735m^4} + \frac{232}{2835m^6} + \cdots$$

surprisingly good also for small mass (we need data from all finite s)

$$\tilde{\Gamma}^S_{WKB(1+2+3)}(m=0) = 0.158054$$

$$\tilde{\Gamma}^S_{\text{exact}}(m=0) = \alpha(1/2) = 0.145873\ldots$$

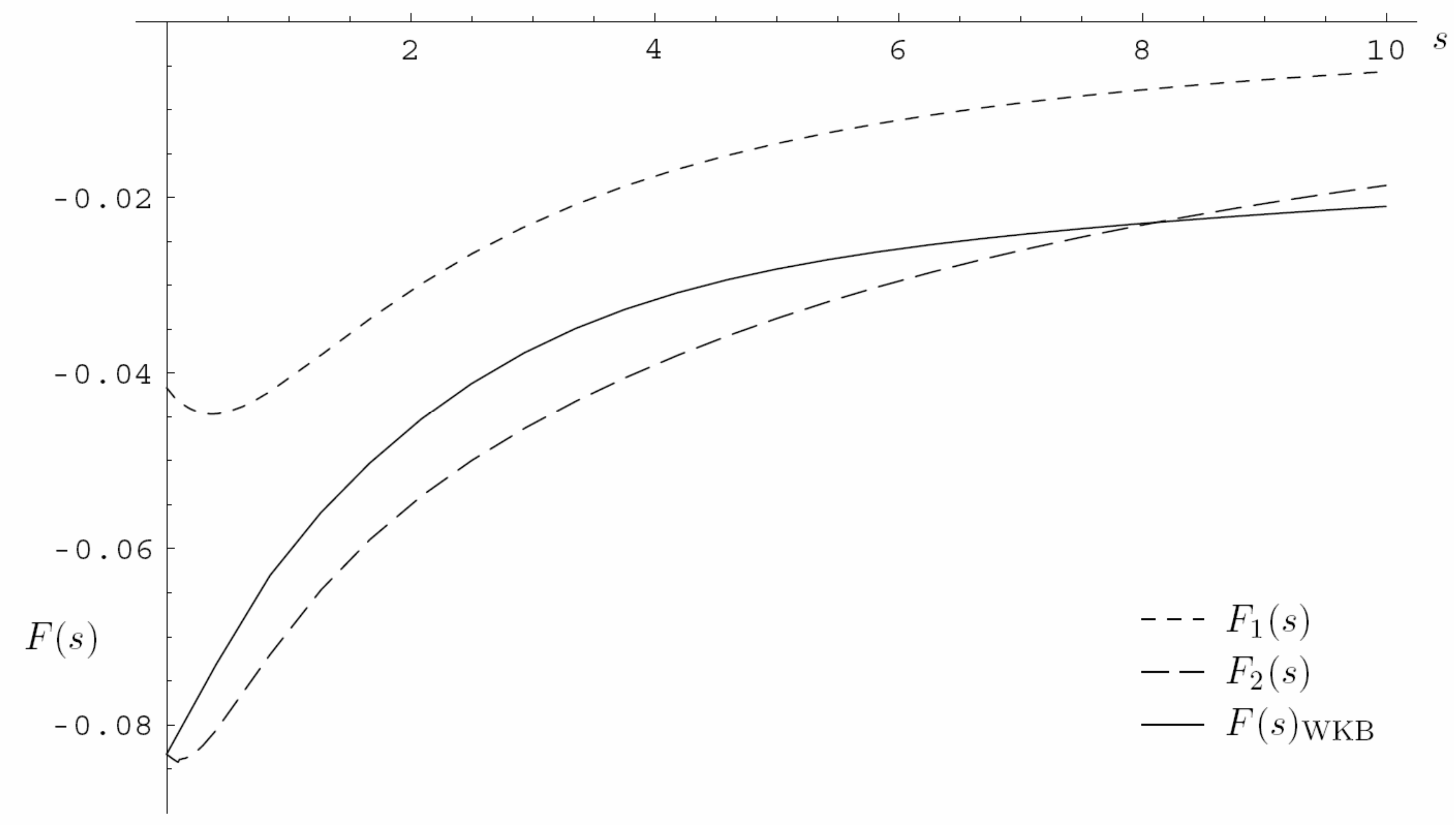
Higher order WKB(2)

$$\eta^{(3)} = \frac{1}{2}\left[\oint\left(\frac{1}{768}\frac{Q^{(4)}(x)}{Q(x)^{5/2}} - \frac{7}{1536}\frac{[Q''(x)]^2}{Q(x)^{7/2}}\,dx\right) - (\text{‘free’})\right]$$

$$\eta^{(3)}_{l,j} = \frac{1}{2}\oint\left\{-\frac{5}{128r^4}\frac{1}{(k^2-\tilde{V}_{(l,j)}(r))^{3/2}} - \frac{1}{128r^2}\frac{d^2V_{(l,j)}(r)}{dr^2}\frac{1}{(k^2-\tilde{V}_{(l,j)}(r))^{5/2}} - \frac{7}{1536}\left(\frac{d^2\tilde{V}_{(l,j)}(r)}{dr^2}\right)^2\frac{1}{(k^2-\tilde{V}_{(l,j)}(r))^{7/2}} - \frac{1}{768}\frac{d^4\tilde{V}_{(l,j)}(r)}{dr^4}\frac{1}{(k^2-\tilde{V}_{(l,j)}(r))^{5/2}}\right\}dr$$

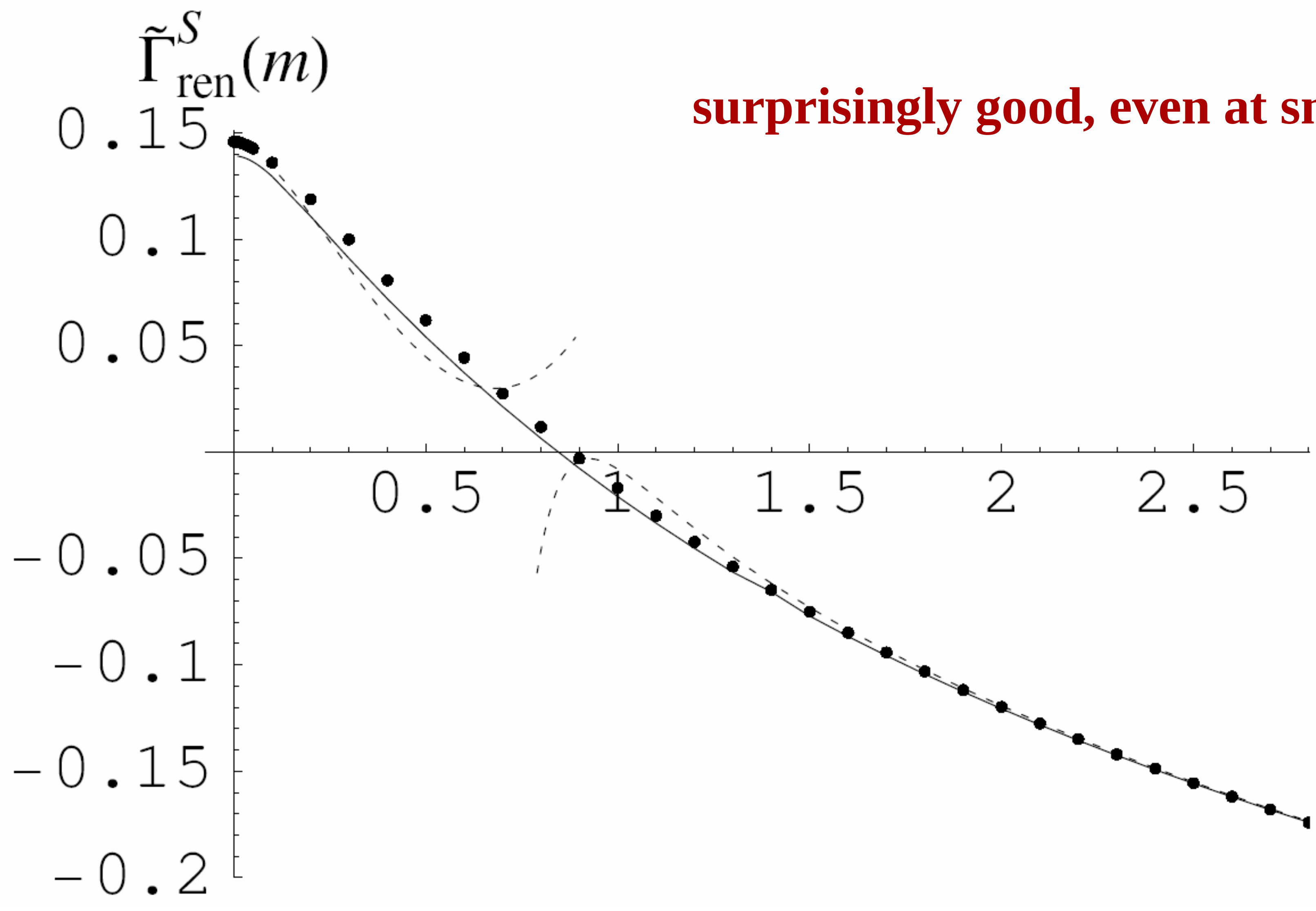
$$f^{(3)}_{(l,j)}(s,r) = \frac{e^{-s\tilde{V}_{(l,j)}(r)}}{2\sqrt{\pi}\sqrt{s}}\left\{\frac{5s^2}{32r^4} - \frac{s^3}{48r^2}\frac{d^2\tilde{V}_{(l,j)}(r)}{dr^2} + \frac{7s^4}{1440}\left(\frac{d^2\tilde{V}_{(l,j)}(r)}{dr^2}\right)^2 - \frac{s^3}{288}\frac{d^4\tilde{V}_{(l,j)}(r)}{dr^4}\right\} - \frac{e^{-s\tilde{V}_l(r)}}{2\sqrt{\pi}\sqrt{s}}\left\{\frac{5s^2}{32r^4} - \frac{s^3}{48r^2}\frac{d^2\tilde{V}_l(r)}{dr^2} + \frac{7s^4}{1440}\left(\frac{d^2\tilde{V}_l(r)}{dr^2}\right)^2 - \frac{s^3}{288}\frac{d^4\tilde{V}_l(r)}{dr^4}\right\}$$

Plot of F(s)



Plot of WKB effective action

surprisingly good, even at small m



Modified Pade approximation

simple analytic interpolation expression for the effective action

$$\tilde{\Gamma}^{S}_{\rm ren}(m) \sim -\frac{1}{6}\ln m + \frac{\frac{1}{6}\ln m + \alpha - (3\alpha+\beta)m^2 - \frac{1}{5}m^4}{1 - 3m^2 + 20m^4 + 15m^6}$$

$$\alpha \equiv \alpha(1/2) \sim 0.145873 \qquad \beta = \tfrac{1}{2}(\ln 2 - \gamma) \sim 0.05797$$

$$-\frac{\ln m}{6} - \frac{1}{75m^2} - \frac{17}{735m^4} + \frac{232}{2835m^6} - \frac{7916}{148225m^8} + \cdots$$

Best fit higher order formula

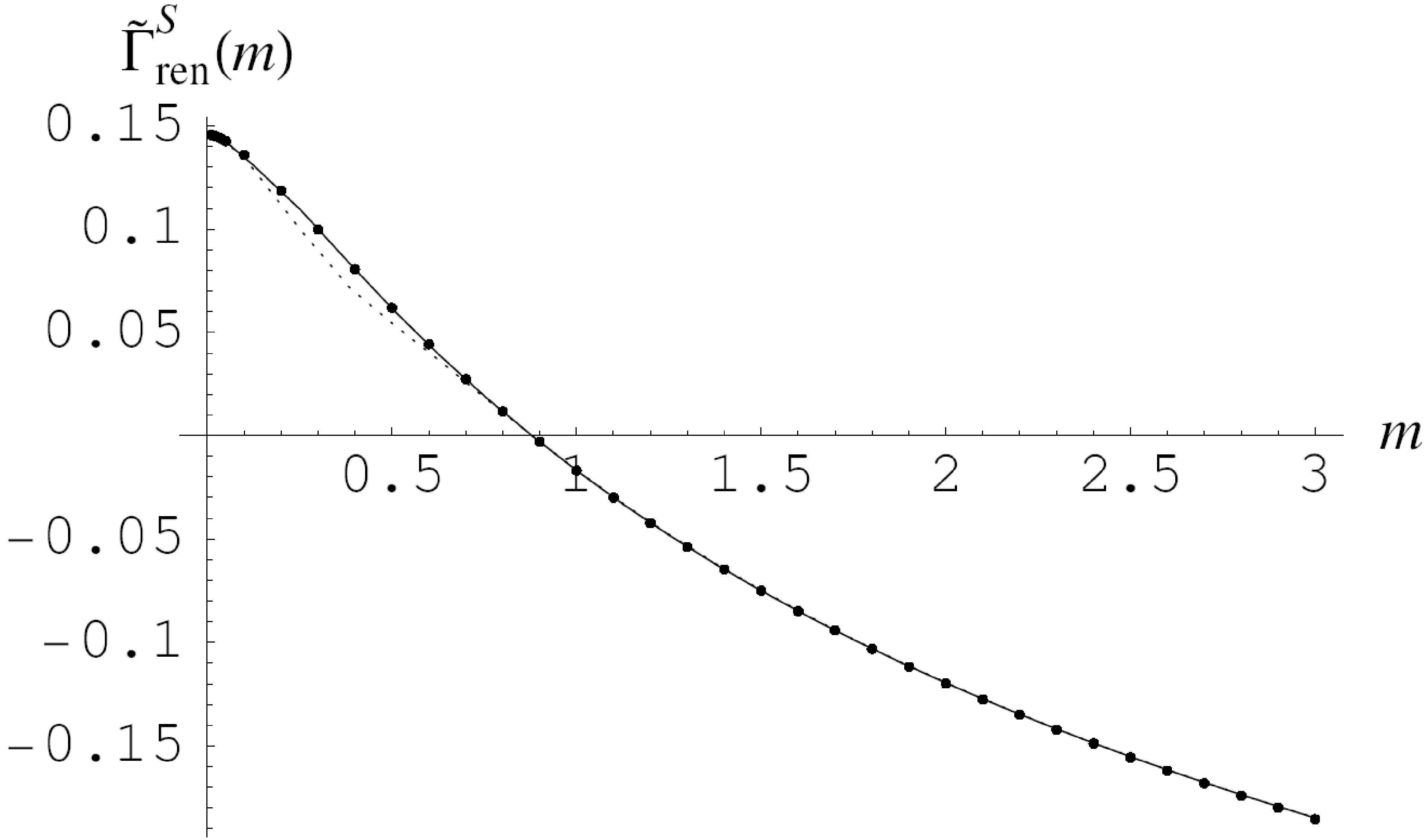
$$\tilde{\Gamma}^{S}_{\rm ren}(m) \sim -\frac{1}{6}\ln m + \frac{\frac{1}{6}\ln m + \alpha - (3\alpha+\beta)m^2 + A1m^4 - A2m^6}{1 - 3m^2 + B1m^4 + B2m^6 + B3m^8}$$

$$B1 = 25(592955/21609A2 + 255/49A1 + 9\alpha + 6\beta),$$

$$B2 = -75(85/49A2 + A1), \quad B3 = 15A2,$$

$$A1 = -13.4138, \quad A2 = 2.64587$$ ← **Best fit**

Plot of Pade approximants



Comparison with derivative expansion

- exact effective action known for covariantly constant $F_{\mu\nu}$

$$F^a_{\mu\nu} = n^a \, F_{\mu\nu} \qquad\qquad \frac{1}{4} F_{\mu\nu} F_{\mu\nu} = f^2$$

- effective action as a constant effective Lagrangian

$$\mathcal{L}^S = -\mathrm{tr} \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{(4\pi T)^2} \left[\left(\frac{f T n^a T^a}{\sinh(f T n^a T^a)} \right)^2 - 1 + \frac{(f T n^a T^a)^2}{3} \right]$$

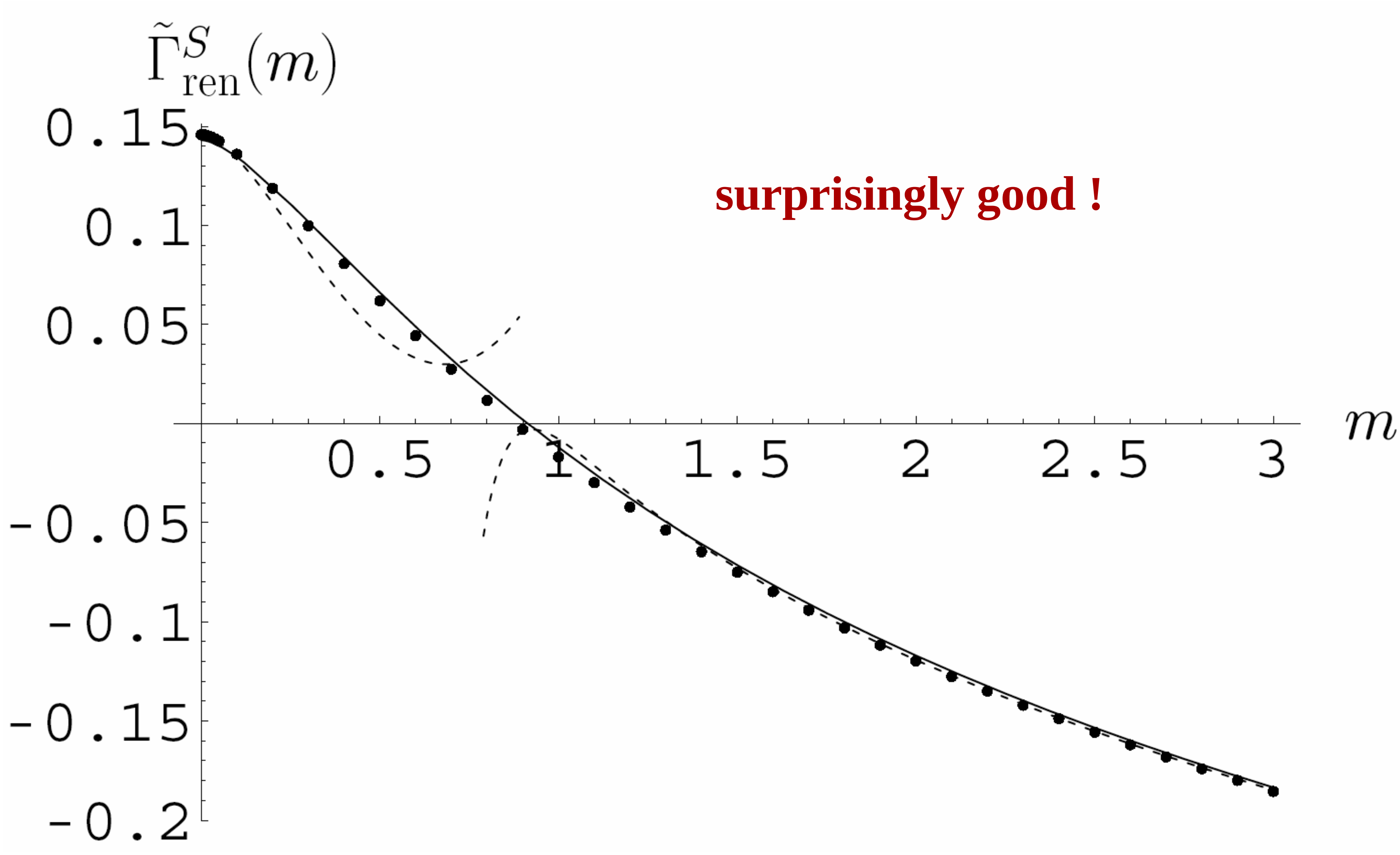
- replace $f^2 = \frac{1}{4} F_{\mu\nu} F_{\mu\nu}$ by **local** expression and integrate

$$\left.\tilde{\Gamma}^S_{\mathrm{ren}}\right]_{\mathrm{DE}} = -\frac{1}{14} \int_0^\infty \frac{dx \, x}{e^{2\pi x} - 1} \left\{ -84 + 14 \ln\left(1 + \frac{48 x^2}{m^4}\right) + 7\sqrt{3} \, \frac{m^2}{x} \arctan\left(\frac{4\sqrt{3}\, x}{m^2}\right) \right.$$

$$\left. + 768 \frac{x^2}{m^4} \; {}_2F_1\left(1, \frac{7}{4}, \frac{11}{4}; -\frac{48 x^2}{m^4}\right) \right\} - \frac{1}{6} \ln m$$

PLB 2004

Comparison with derivative expansion



conclusions

- successful application of determinant theorem to **radial** partial differential operator

- excellent results for **instanton** background

- regularization/renormalization approach is **very general**

- **WKB** approximations:
  good for small mass as well as for large mass

- methods are suitable to various computations:
  -tunneling amplitude, ground state energy,
  quantum effects to solitons

- **derivative expansion** surprisingly accurate