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Vacuum energy in radial backgrounds

Klaus Kirsten

Baylor University
Department of Mathematics
Waco, TX 76798
USA

KK and G.V. Dunne, in preparation

- Finite temperature free field theory:

Let $\mathcal{M} = S^1 \times \mathbb{R}^3$.

$$\begin{aligned} P\phi_\omega(\tau,\vec{x}) &= \left(-\frac{\partial^2}{\partial\tau^2} + P_S\right)\phi_\omega(\tau,\vec{x}) \\ &= \left(-\frac{\partial^2}{\partial\tau^2} - \Delta + m^2 + V(r)\right)\phi_\omega(\tau,\vec{x}) \\ &= \omega^2\phi_\omega(\tau,\vec{x}) \end{aligned}$$

- Ansatz for eigenfunctions:

$$\phi_{n,k}(\tau,\vec{x}) = \frac{1}{\beta}\, e^{\frac{2\pi i n}{\beta}\tau}\psi_k(r,\Omega)$$

- Eigenvalues:

$$\omega^2 = \left(\frac{2\pi n}{\beta}\right)^2 + m^2 + k^2, \qquad n \in \mathbb{Z}$$

$$(-\Delta + V(r))\psi_k(r,\Omega) = k^2\psi_k(r,\Omega).$$

- Associated zeta function:

$$\zeta_P(s) = \sum_{n=-\infty}^{\infty}\sum_k \left[\left(\frac{2\pi n}{\beta}\right)^2 + m^2 + k^2\right]^{-s}, \quad \Re s > 2$$

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- Casimir energy:

$$E = -\frac{1}{2} \frac{\partial}{\partial \beta} \zeta'_{P/\mu^2}(0)$$

- Zero temperature:

$$\begin{aligned} E_{Cas} &= \lim_{\beta \to \infty} E \\ &= \frac{1}{2} \, \mathrm{FP} \; \zeta_{P_S} \left(-\frac{1}{2} \right) - \frac{1}{4\sqrt{\pi}} a_2(P_S) \ln \left(\frac{\mu e}{2} \right) \end{aligned}$$

where

$$\begin{aligned} \zeta_{P_S}(s) &= \sum_k (m^2 + k^2)^{-s}, \quad \Re s > \frac{3}{2} \\ K_{P_S}(t) &= \sum_k e^{-t(m^2+k^2)} \sim \sum_{l=0}^{\infty} a_l(P_S) \, t^{l-3/2} \end{aligned}$$

- Analysis of radial eigenvalue problem:

$$\psi_{k,l,m} = \frac{1}{r}\phi_{k,l}(r) Y_{lm}(\theta,\varphi) \quad \Rightarrow$$

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - V(r) + k^2\right)\phi_{k,l}(r) = 0$$

- Regular solution:

$$\phi_{k,l}(r) \;\sim\; \hat{j}_l(kr) = \sqrt{\frac{\pi kr}{2}} J_{l+1/2}(kr) \quad \text{as } r \to 0$$

$$\phi_{k,l}(r) \;\sim\; \frac{i}{2}\left[f_l(k)\hat{h}_l^-(kr) - f_l^*(k)\hat{h}_l^+(kr)\right] \quad \text{as } r \to \infty$$

- System in a finite ball of radius R:

implicit eigenvalue equation

$$f_l(k)\hat{h}_l^-(kR) - f_l^*(k)\hat{h}_l^+(kR) = 0$$

- Zeta function analysis:

$$\zeta_P(s) = \sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty}(2l+1)\times$$

$$\int_\gamma \frac{dk}{2\pi i}(k^2+m^2+p_n^2)^{-s}\frac{\partial}{\partial k}\ln\left[f_l(k)\,\hat{h}_l^-(kR) - f_l^*(k)\,\hat{h}_l^+(kR)\right]$$

- Minkowski space theory subtracted:

$$\zeta_P(s) = \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} (2l+1) \times$$

$$\int_\gamma \frac{dk}{2\pi i} (k^2+m^2+p_n^2)^{-s} \frac{\partial}{\partial k} \ln \left[\frac{f_l(k)\hat{h}_l^-(kR) - f_l^*(k)\hat{h}_l^+(kR)}{\hat{h}_l^-(kR) - \hat{h}_l^+(kR)} \right]$$

$$+ \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} (2l+1) \sum_j (m^2+p_n^2-\kappa_{j,l}^2)^{-s}$$

$$(R\to\infty) = \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} (2l+1) \int_\gamma \frac{dk}{2\pi i} (k^2+m^2+p_n^2)^{-s} \frac{\partial}{\partial k} \ln f_l(k)$$

$$+ \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} (2l+1) \sum_j (m^2+p_n^2-\kappa_{j,l}^2)^{-s}$$

$$= \frac{\sin \pi s}{\pi} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} (2l+1) \int_{\sqrt{p_n^2+m^2}}^{\infty} dk\ (k^2-m^2-p_n^2)^{-s} \frac{\partial}{\partial k} \ln f_l(ik)$$

- Asymptotics of the Jost function:
  Regular solution:

$$\phi_{k,l}(r) = \hat{j}_l(kr) + \int_0^r dr' \mathcal{G}_{k,l}(r,r') V(r') \phi_{k,l}(r')$$

where

$$\mathcal{G}_{k,l}(r,r') = \frac{i}{2k} \left[\hat{h}_l^-(kr) \hat{h}_l^+(kr') - \hat{h}_l^+(kr) \hat{h}_l^-(kr') \right]$$

- Asymptotically as $r \to \infty$:

$$\begin{aligned} \phi_{k,l}(r) \quad &\sim \quad \frac{i}{2} \left\{ \left[1 + \frac{1}{k} \int_0^\infty dr' \; \hat{h}_l^+(kr') V(r') \phi_{k,l}(r') \right] \hat{h}_l^-(kr) \right. \\ &\quad \left. - \left[1 + \frac{1}{k} \int_0^\infty dr' \; \hat{h}_l^+(kr') V(r') \phi_{k,l}(r') \right] \hat{h}_l^+(kr) \right\} \end{aligned}$$

- Jost function ($\nu = l + 1/2$):

$$\begin{aligned} f_l(ik) \quad &= \quad 1 + \int_0^\infty dr \; r V(r) \phi_{ik,l}(r) K_\nu(kr) \\ \ln f_l(ik) \quad &= \quad \int_0^\infty dr \; r V(r) K_\nu(kr) I_\nu(kr) \\ &\quad - \int_0^\infty dr \; r V(r) K_\nu^2(kr) \int_0^r dr' \; r' V(r') I_\nu^2(kr') \; + \mathcal{O}(V^3) \end{aligned}$$

- Use Bessel function asymptotics:

$$\begin{aligned}
\ln f_l^{asym}(ik) &= \frac{1}{2\nu}\int_0^\infty dr\, \frac{rV(r)}{\left(1+\frac{k^2r^2}{\nu^2}\right)^{1/2}} \\
&+\frac{1}{16\nu^3}\int_0^\infty dr\, \frac{rV(r)}{\left(1+\frac{k^2r^2}{\nu^2}\right)^{3/2}}\left[1-\frac{6}{1+\frac{k^2r^2}{\nu^2}}+\frac{5}{\left(1+\frac{k^2r^2}{\nu^2}\right)^2}\right] \\
&-\frac{1}{8\nu^3}\int_0^\infty dr\, \frac{r^3V^2(r)}{\left(1+\frac{k^2r^2}{\nu^2}\right)^{3/2}}
\end{aligned}$$

- Split zeta function:

$$\zeta_P(s) = \zeta_f(s) + \zeta_{as}(s)$$

where

$$\begin{aligned}
\zeta_f(s) &= \frac{\sin \pi s}{\pi}\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty}(2l+1)\times \\
&\int_{\sqrt{p_n^2+m^2}}^{\infty} dk\ (k^2-p_n^2-m^2)^{-s}\frac{\partial}{\partial k}\ln\left[f_l(ik)-f_l^{asym}(ik)\right]
\end{aligned}$$

$$\begin{aligned}
\zeta_{as}(s) &= \frac{\sin \pi s}{\pi}\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty}(2l+1)\times \\
&\int_{\sqrt{p_n^2+m^2}}^{\infty} dk\ (k^2-p_n^2-m^2)^{-s}\frac{\partial}{\partial k}\ln f_l^{asym}(ik)
\end{aligned}$$

- Zeta determinant:

$$\zeta_f'(0) = -\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty}(2l+1)\left[\ln f_l(i\sqrt{p_n^2+m^2}) - \ln f_l^{asym}(i\sqrt{p_n^2+m^2})\right]$$

- Relevant factor in $\ln f_l^{asym}$:

$$1+\frac{(p_n^2+m^2)r^2}{\nu^2} = \left(1+\frac{p_n^2\ r^2}{\nu^2}\right)\left(1+\frac{m^2r^2}{\nu^2+p_n^2\ r^2}\right)$$

- Rewrite asymptotics:

$$\begin{aligned}
\ln f_l^{asym}(i\sqrt{p_n^2+m^2}) &= \frac{1}{2\nu}\int_0^\infty dr \frac{rV(r)}{\left(1+\frac{(p_n^2+k^2)r^2}{\nu^2}\right)^{1/2}} + \dots \\
&= \frac{1}{2\nu}\int_0^\infty dr \frac{rV(r)}{\left(1+\frac{p_n^2\ r^2}{\nu^2}\right)^{1/2}} \times \\
&\quad \left[\frac{1}{\left(1+\frac{m^2r^2}{\nu^2+p_n^2\ r^2}\right)^{1/2}} - 1 - \frac{1}{2}\ \frac{m^2r^2}{\nu^2+p_n^2\ r^2}\right] \\
&\quad +\frac{1}{2\nu}\int_0^\infty dr \frac{rV(r)}{\left(1+\frac{p_n^2\ r^2}{\nu^2}\right)^{1/2}} \\
&\quad +\frac{1}{2\nu}\int_0^\infty dr \frac{rV(r)}{\left(1+\frac{p_n^2\ r^2}{\nu^2}\right)^{1/2}} \cdot \frac{1}{2}\ \frac{m^2r^2}{\nu^2+p_n^2\ r^2}
\end{aligned}$$

- Apply same procedure to $\zeta_{as}(s)$

Final answer at $T = 0$ in four dimensions

$$
\begin{aligned}
\zeta_P'(0) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} dp \sum_{l=0}^{\infty}(2l+1)\left[\ln f_l(i\sqrt{p^2+m^2})\right. \\
&\quad -\frac{1}{2}\int_0^\infty dr \frac{rV(r)}{(\nu^2+p^2\,r^2)^{1/2}} + \frac{1}{8}\int_0^\infty dr \frac{r^3V(r)(V(r)+2m^2)}{(\nu^2+p^2\,r^2)^{3/2}} \\
&\quad \left. -\frac{1}{16}\int_0^\infty dr \frac{rV(r)}{(\nu^2+p^2\,r^2)^{3/2}}\left(1-\frac{6\nu^2}{\nu^2+p^2\,r^2}+\frac{5\nu^4}{(\nu^2+p^2\,r^2)^2}\right)\right] \\
&\quad +\frac{1}{24\pi}\left[\gamma + 3\ln 2 + 12\zeta_R'(-1)\right]\cdot\int_0^\infty drV(r) \\
&\quad +\frac{1}{4\pi}\int_0^\infty drV(r)(V(r)+2m^2)r^2(\gamma+\ln[4r])
\end{aligned}
$$