Exact Multiple Scattering Results
Weak-Coupling Forces Between Bodies

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I. Multiple Scattering Technique

The multiple scattering approach starts from the well-known formula for the vacuum energy or Casimir energy (for simplicity here we first restrict attention to a massless scalar field)(τ is the "infinite" time that the configuration exists) [Schwinger, 1975]

$$E = \frac{i}{2\tau}\mathrm{Tr}\ln G \to \frac{i}{2\tau}\mathrm{Tr}\ln G G_0^{-1},$$

where G (G_0) is the Green's function,

$$(-\partial^2 + V)G = 1, \quad +\mathrm{BC}, \quad -\partial^2 G_0 = 1.$$

T-matrix

Now we define the T-matrix,

$$T = S - 1 = V(1 + G_0 V)^{-1}.$$

If the potential has two disjoint parts,
$V = V_1 + V_2$ it is easy to derive the interaction between the two bodies (potentials):

$$\begin{aligned} E_{12} &= -\frac{i}{2\tau} \mathrm{Tr} \ln(1 - G_0 T_1 G_0 T_2) \\ &= -\frac{i}{2\tau} \mathrm{Tr} \ln(1 - V_1 G_1 V_2 G_2), \end{aligned}$$

where $G_i = (1 + G_0 V_i)^{-1} G_0, \quad i = 1, 2.$

Multipole expansion

To proceed to apply this method to general bodies, we use an even older technique, the multipole expansion. Let's illustrate this with a $2+1$ dimensional version, which allows us to describe cylinders with parallel axes. We seek an expansion of the free Green's function

$$G_0(\mathbf{R}+\mathbf{r}'-\mathbf{r}) = \frac{e^{i|\omega||\mathbf{r}-\mathbf{R}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{R}-\mathbf{r}'|}$$

$$= \int \frac{dk_z}{2\pi} e^{ik_z(z-Z-z')} g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp),$$

Reduced Green's function

$$g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp) = \int \frac{(d^2k_\perp)}{(2\pi)^2} \frac{e^{-i\mathbf{k}_\perp\cdot\mathbf{R}_\perp} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}_\perp - \mathbf{r}'_\perp)}}{k_\perp^2 + k_z^2 + \zeta^2}.$$

As long as the two potentials do not overlap, so that we have $\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp \neq 0$, we can write an expansion in terms of modified Bessel functions:

$$g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp) = \sum_{m,m'} I_m(\kappa r) e^{im\phi} I'_m(\kappa r') e^{-im'\phi'}$$

$$\times \tilde{g}^0_{m,m'}(\kappa R), \quad \kappa^2 = k_z^2 + \zeta^2.$$

Expression for $g^0_{m,m'}$

By Fourier transforming, and using the definition of the Bessel function

$$i^m J_m(kr) = \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{-im\phi} e^{ikr\cos\phi},$$

we easily find

$$\tilde{g}^0_{m,m'}(\kappa R) = \frac{1}{2\pi} \int \frac{dk\, k}{k^2+\kappa^2} J_{m-m'}(kR) \frac{J_m(kr) J_m(kr')}{I_m(\kappa r) I_m(\kappa r')}$$
$$= \frac{(-1)^{m'}}{2\pi} K_{m-m'}(\kappa R).$$

Discrete matrix realization

Thus we can derive an expression for the interaction between two bodies, in terms of discrete matrices,

$$\mathfrak{E} \equiv \frac{E_{\text{int}}}{L} = \frac{1}{8\pi^2} \int d\zeta \, dk_z \ln \det \left(1 - \tilde{g}^0 \tilde{T}_1 \tilde{g}^{0\top} \tilde{T}_2 \right),$$

where the $\tilde{T}$ matrix elements are given by

$$\tilde{T}_{mm'} = \int dr\, r\, d\phi \int dr'\, r'\, d\phi' I_m(\kappa r) e^{-im\phi} I_{m'}(\kappa r') e^{im'\phi'} \times T(r,\phi;r',\phi').$$

Partial list of recent references

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Interaction between cylinders

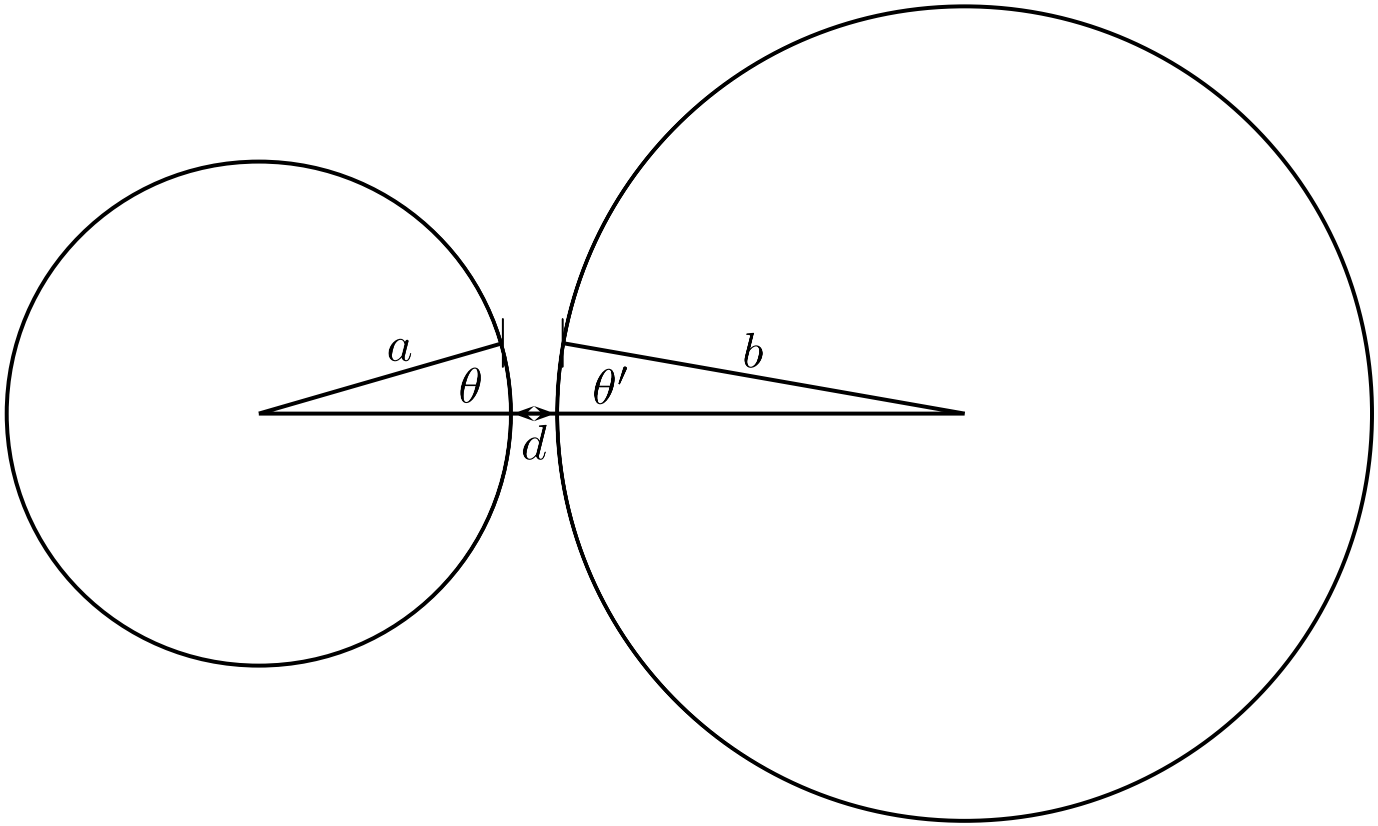


Figure 1: Geometry of two cylinders (or two spheres) with radii a and b, respectively, and distances between their centers of $R > a + b$.

Semitransparent cylinders

Consider two parallel semitransparent cylinders, of radii a and b, respectively, lying outside each other, described by the potentials

$$V_1 = \lambda_1 \delta(r-a), \quad V_2 = \lambda_2 \delta(r'-b),$$

with the separation between the centers R satisfying $R > a+b$. It is easy to work out the scattering matrix in this situation,

$$(t_1)_{mm'} = 2\pi\lambda_1 a \delta_{mm'} \frac{I_m^2(\kappa a)}{1+\lambda_1 a I_m(\kappa a) K_m(\kappa a)}.$$

Cylinder interaction

Thus the Casimir energy per unit length is

$$\mathfrak{E} = \frac{1}{4\pi}\int_0^\infty d\kappa\,\kappa\,\mathbf{tr}\ln(1-A),$$

where $A = B(a)B(b)$, in terms of the matrices

$$B_{mm'}(a) = K_{m+m'}(\kappa R)\frac{\lambda_1 a I_{m'}^2(\kappa a)}{1+\lambda_1 a I_{m'}(\kappa a)K_{m'}(\kappa a)}.$$

Weak-coupling

In weak coupling, the formula for the interaction energy between two cylinders is

$$\mathfrak{E} = -\frac{\lambda_1\lambda_2 ab}{4\pi R^2} \sum_{m,m'=-\infty}^{\infty} \int_0^\infty dx\, x\, K^2_{m+m'}(x)$$
$$\times I^2_m(xa/R) I^2_{m'}(xb/R).$$

Power series expansion

One merely exploits the small argument expansion for the modified Bessel functions $I_m(xa/R)$ and $I_{m'}(xb/R)$:

$$I_m^2(x) = \left(\frac{x}{2}\right)^{2|m|} \sum_{n=0}^{\infty} Z_{|m|,n} \left(\frac{x}{2}\right)^{2n},$$

where the coefficients $Z_{m,n}$ are

$$Z_{m,n} = \frac{2^{2(m+n)}\,\Gamma\left(m+n+\frac{1}{2}\right)}{\sqrt{\pi}\; n!\;(2m+n)!\;\Gamma(m+n+1)}.$$

Closed form result

In this case we get an amazingly simple result

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{4\pi R^2} \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{a}{R}\right)^{2n} P_n(\mu),$$

where $\mu = b/a$, and where by inspection we identify the binomial coefficients

$$P_n(\mu) = \sum_{k=0}^{n} \binom{n}{k}^2 \mu^{2k}.$$

Closed form result (cont.)

Remarkably, it is possible to perform the sums, so we obtain the following closed form for the interaction between two weakly-coupled cylinders:

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{8\pi R^2}\left[\left(1-\left(\frac{a+b}{R}\right)^2\right)\left(1-\left(\frac{a-b}{R}\right)^2\right)\right]^{-1/2}.$$

PFA

We note that in the limit $R - a - b = d \to 0$, d being the distance between the closest points on the two cylinders, we recover the proximity force theorem in this case

$$U(d) = -\frac{\lambda_1\lambda_2}{32\pi}\sqrt{\frac{2ab}{R}}\frac{1}{d^{1/2}}, \quad d \ll a, b.$$

The rate of approach is given by

$$\frac{\mathfrak{E}}{U} \approx 1 - \frac{1+\mu+\mu^2}{4\mu}\frac{d}{R} \approx 1 - \frac{R^2 - aR + a^2}{4a(R-a)}\frac{d}{R}.$$

$a = b$

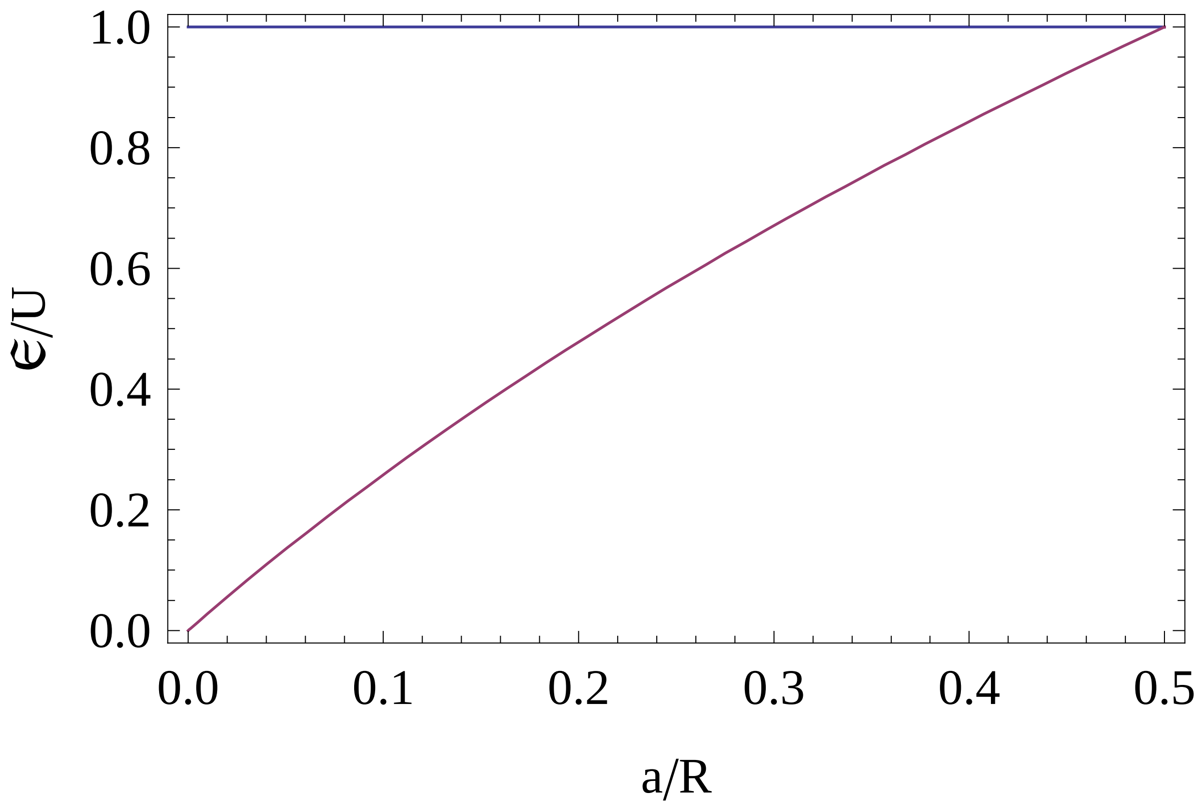


Figure 2: Plotted is the ratio of the exact interaction energy of two weakly-coupled cylinders to the proximity force approximation

$b/a = 99$

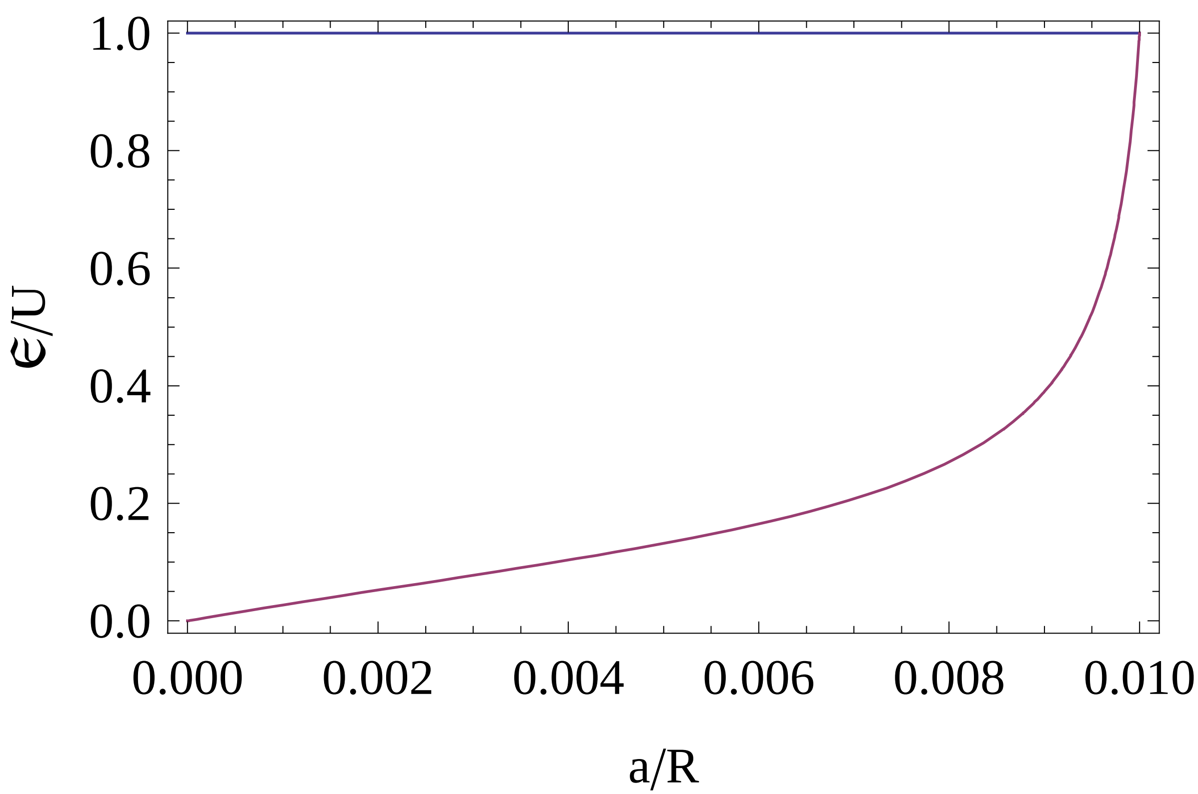


Figure 3: Plotted is the ratio of the exact interaction energy of two weakly-coupled cylinders to the proximity force approximation

Cylinder/plane interaction

By the method of images, we can find the interaction between semitransparent cylinder and a Dirichlet plane is

$$\mathfrak{E} = \frac{1}{4\pi}\int_0^\infty \kappa\, d\kappa\, \mathrm{tr}\, \ln(1 - B(a)),$$

where $B(a)$ is given above. In the strong-coupling limit this result agrees with that given by Bordag, because

$$\mathrm{tr}\, B^s = \mathrm{tr}\, \tilde{B}^s, \quad \tilde{B}_{mm'} = \frac{1}{K_m(\kappa a)} K_{m+m'}(\kappa R) I_{m'}(\kappa a).$$

Exact cylinder/plane energy

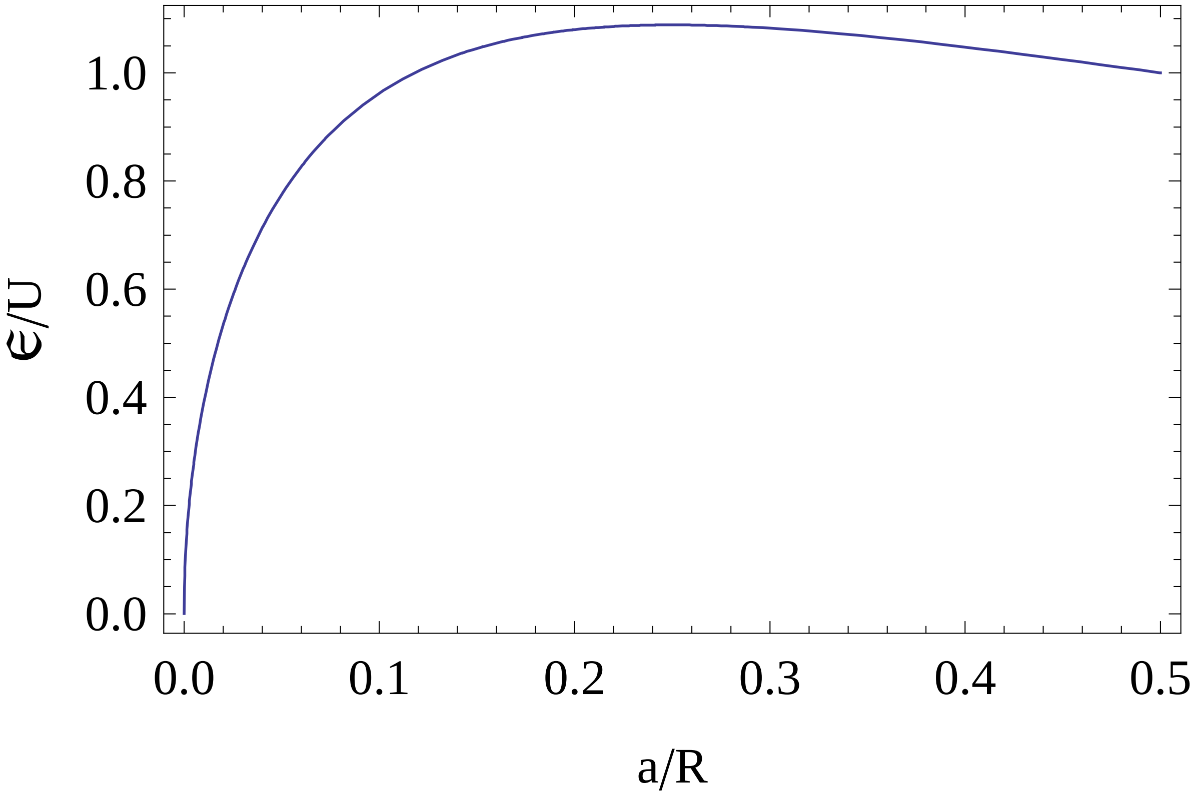
In exactly the same way, we can obtain a closed-form result for the interaction energy between a Dirichlet plane and a weakly-coupled cylinder of radius a separated by a distance $R/2$. The result is again quite simple:

$$\mathfrak{E} = -\frac{\lambda a}{4\pi R^2}\left[1 - \left(\frac{2a}{R}\right)^2\right]^{-3/2}.$$

In the limit as $d \to 0$, this agrees with the PFA:

$$U(d) = -\frac{\lambda}{64\pi}\frac{\sqrt{2a}}{d^{3/2}}.$$

Comparison of PFA and exact



3-dimensional formalism

The three-dimensional formalism is very similar. In this case, the free Green's function has the representation

$$G_0(\mathbf{R}+\mathbf{r}'-\mathbf{r}) = \sum_{lm,l'm'} j_l(i|\zeta|r) j_{l'}(i|\zeta|r') Y^*_{lm}(\hat{\mathbf{r}}) Y_{l'm'}(\hat{\mathbf{r}}') \times g_{lm,l'm'}(\mathbf{R}).$$

Reduced Green's function

The reduced Green's function can be written in the form

$$g^0_{lm,l'm'}(\mathbf{R}) = (4\pi)^2 i^{l'-l} \int \frac{(d\mathbf{k})}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{k^2+\zeta^2} \frac{j_l(kr)j_{l'}(kr')}{j_l(i|\zeta|r)j_{l'}(i|\zeta|r')}$$

$$\times Y_{lm}(\hat{\mathbf{k}})Y^*_{l'm'}(\hat{\mathbf{k}}).$$

Now we use the plane-wave expansion once again, this time for $e^{i\mathbf{k}\cdot\mathbf{R}}$,

$$e^{i\mathbf{k}\cdot\mathbf{R}} = 4\pi \sum_{l''m''} i^{l''} j_{l''}(kR) Y_{l''m''}(\hat{\mathbf{R}}) Y^*_{l''m''}(\hat{\mathbf{k}}),$$

so now we encounter something new, an integral over three spherical harmonics,

$$\int d\hat{\mathbf{k}} Y_{lm}(\hat{\mathbf{k}}) Y^*_{l'm'}(\hat{\mathbf{k}}) Y^*_{l''m''}(\hat{\mathbf{k}}) = C_{lm,l'm',l''m''},$$

Wigner coefficients

where

$$C_{lm,l'm',l''m''} = (-1)^{m'+m''}\sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}} \times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}.$$

The three-j symbols (Wigner coefficients) here vanish unless $l + l' + l''$ is even.

Reduced Green's function

This fact is crucial, since because of it we can follow the previous method of writing $j_{l''}(kR)$ in terms of Hankel functions of the first and second kind, using the reflection property of the latter, $h^{(2)}_{l''}(kR) = (-1)^{l''} h^{(1)}_{l''}(-kR)$, and then extending the k integral over the entire real axis to a contour integral closed in the upper half plane.

$$g^0_{lm,l'm'}(\mathbf{R}) = 4\pi i^{l'-l}\sqrt{\frac{2|\zeta|}{\pi R}}\sum_{l''m''} C_{lm,l'm',l''m''}$$

$$\times K_{l''+1/2}(|\zeta|R) Y_{l''m''}(\hat{\mathbf{R}}).$$

Casimir interaction of spheres

For the case of two semitransparent spheres that are totally outside each other,

$$V_1(r) = \lambda_1 \delta(r-a), \quad V_2(r') = \lambda_2 \delta(r'-b),$$

in terms of spherical coordinates centered on each sphere, it is again very easy to calculate the scattering matrices,

$$T_1(\mathbf{r}, \mathbf{r}') = \frac{\lambda_1}{a^2} \delta(r-a) \delta(r'-a)$$

$$\times \sum_{lm} \frac{Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}}')}{1 + \lambda_1 a K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|a)},$$

Scattering matrix element

and then the harmonic transform is very similar to that seen for the cylinder, ($k = i|\zeta|$)

$$(t_1)_{lm,l'm'} = \int (d\mathbf{r})(d\mathbf{r}') j_l(kr) Y^*_{lm}(\hat{\mathbf{r}}) j_{l'}(kr') Y_{l'm'}(\hat{\mathbf{r}}') T_1(\mathbf{r}, \mathbf{r}'$$

$$= \delta_{ll'}\delta_{mm'}(-1)^l \frac{\lambda_1 a\pi}{2|\zeta|} \frac{I^2_{l+1/2}(|\zeta|a)}{1 + \lambda_1 a K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|a)}.$$

Interaction energy

Let us suppose that the two spheres lie along the z-axis, that is, $\mathbf{R} = R\hat{\mathbf{z}}$. Then we can simplify the expression for the energy somewhat by using $Y_{lm}(\theta = 0) = \delta_{m0}\sqrt{(2l+1)/4\pi}$. The formula for the energy of interaction becomes

$$E = \frac{1}{2\pi}\int_0^\infty d\zeta\, \mathrm{tr}\ln(1 - A),$$

where the matrix

$$A_{lm,l'm'} = \delta_{m,m'}\sum_{l''} B_{ll''m}(a)B_{l''l'm}(b)$$

$B_{ll'm}$

$$B_{ll'm}(a) = \frac{\sqrt{\pi}}{\sqrt{2\zeta R}} i^{-l+l'} \sqrt{(2l+1)(2l'+1)} \sum_{l''} (2l''+1)$$

$$\times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} \frac{K_{l''+1/2}(\zeta R) \lambda_1 a I^2_{l'+1/2}(\zeta a)}{1 + \lambda_1 a I_{l'+1/2}(\zeta a) K_{l'+1/2}(\zeta a)}$$

Note that the phase always cancels in the trace.

Weak coupling

For weak coupling, a major simplification results because of the orthogonality property,

$$\sum_{m=-l}^{l} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} l & l' & l''' \\ m & -m & 0 \end{pmatrix} = \delta_{l''l'''} \frac{1}{2l''+1}, l \leq l'.$$

$$E = -\frac{\lambda_1 a \lambda_2 b}{4R} \int_0^\infty \frac{dx}{x} \sum_{ll'l''} (2l+1)(2l'+1)(2l''+1)$$
$$\times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}^2 K^2_{l''+1/2}(x) I^2_{l+1/2}(xa/R) I^2_{l'+1/2}(xb/R).$$

Power series expansion

As with the cylinders, we expand the modified Bessel functions of the first kind in power series in $a/R, b/R < 1$. This expansion yields the infinite series

$$E = -\frac{\lambda_1 a \lambda_2 b}{4\pi R}\frac{ab}{R^2}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{m=0}^{n} D_{n,m}\left(\frac{a}{R}\right)^{2(n-m)}\left(\frac{b}{R}\right)^{2m}$$

where by inspection of the first several $D_{n,m}$ coefficients we can identify them as

$$D_{n,m} = \frac{1}{2}\binom{2n+2}{2m+1},$$

Closed form

and now we can immediately sum the expression for the Casimir interaction energy to give the closed form

$$E = \frac{\lambda_1 a \lambda_2 b}{16\pi R} \ln\left(\frac{1-\left(\frac{a+b}{R}\right)^2}{1-\left(\frac{a-b}{R}\right)^2}\right).$$

PFA

Again, when $d = R - a - b \ll a, b$, the proximity force theorem is reproduced:

$$U(d) \sim \frac{\lambda_1 \lambda_2 ab}{16\pi R} \ln(d/R), \quad d \ll a, b.$$

However, as the figures demonstrate, the approach is not very smooth, even for equal-sized spheres. The ratio of the energy to the PFA is $(b/a = \mu)$

$$\frac{E}{U} = 1 + \frac{\ln[(1+\mu)^2/2\mu]}{\ln d/R}, \quad d \ll a, b.$$

$a = b$; truncation at 100 shown

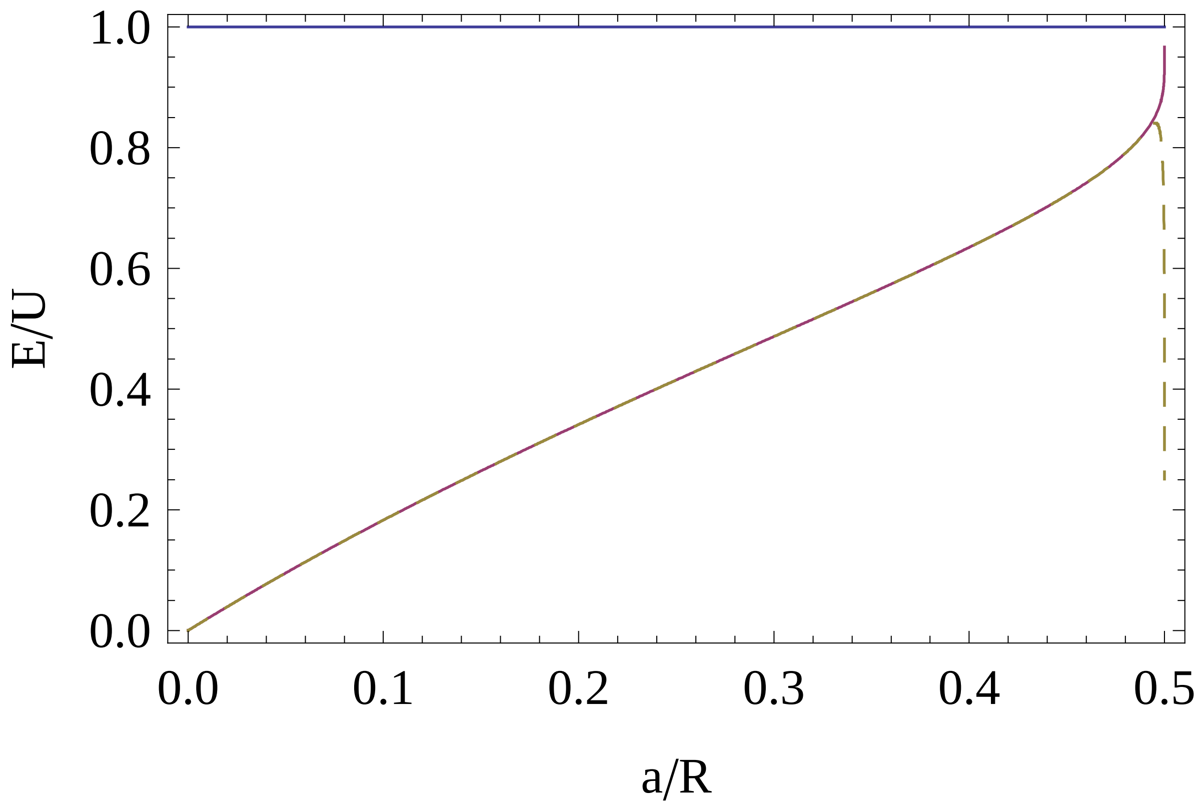


Figure 4: Plotted is the ratio of the exact interaction energy of two weakly-coupled spheres to the proximity force approximation

$b/a = 49$

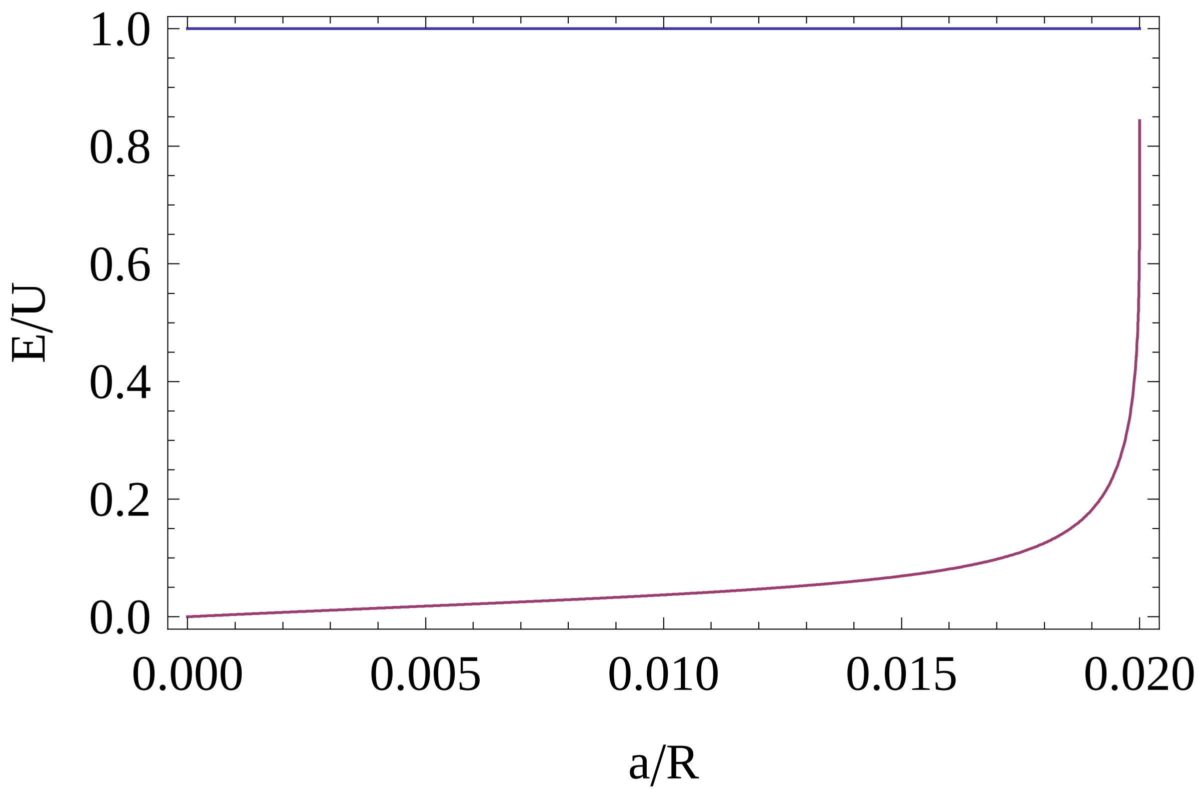


Figure 5: Plotted is the ratio of the exact interaction energy of two weakly-coupled spheres to the proximity force approximation

Exact plane/sphere energy

In just the way indicated above, we can obtain a closed-form result for the interaction energy between a weakly-coupled sphere and a Dirichlet plane. Using the simplification that

$$\sum_{m=-l}^{l}(-1)^m \begin{pmatrix} l & l & l' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} l & l & l' \\ 0 & 0 & 0 \end{pmatrix} = \delta_{l'0},$$

we can write the interaction energy in the form

$$E = -\frac{\lambda a}{2\pi R}\int_0^\infty dx \sum_{l=0}^\infty \sqrt{\frac{\pi}{2x}}(2l+1)K_{1/2}(x)I^2_{l+1/2}\left(x\frac{a}{R}\right)$$

Then in terms of $R/2$ as the distance between the center of the sphere and the plane, the exact interaction energy is

$$E = -\frac{\lambda}{2\pi}\left(\frac{a}{R}\right)^2 \frac{1}{1-(2a/R)^2},$$

which as $a \to R/2$ reproduces the proximity force limit, contained in the (ambiguously defined) PFA formula

$$U = -\frac{\lambda}{8\pi}\frac{a}{d}.$$

Exact energy vs. PFA

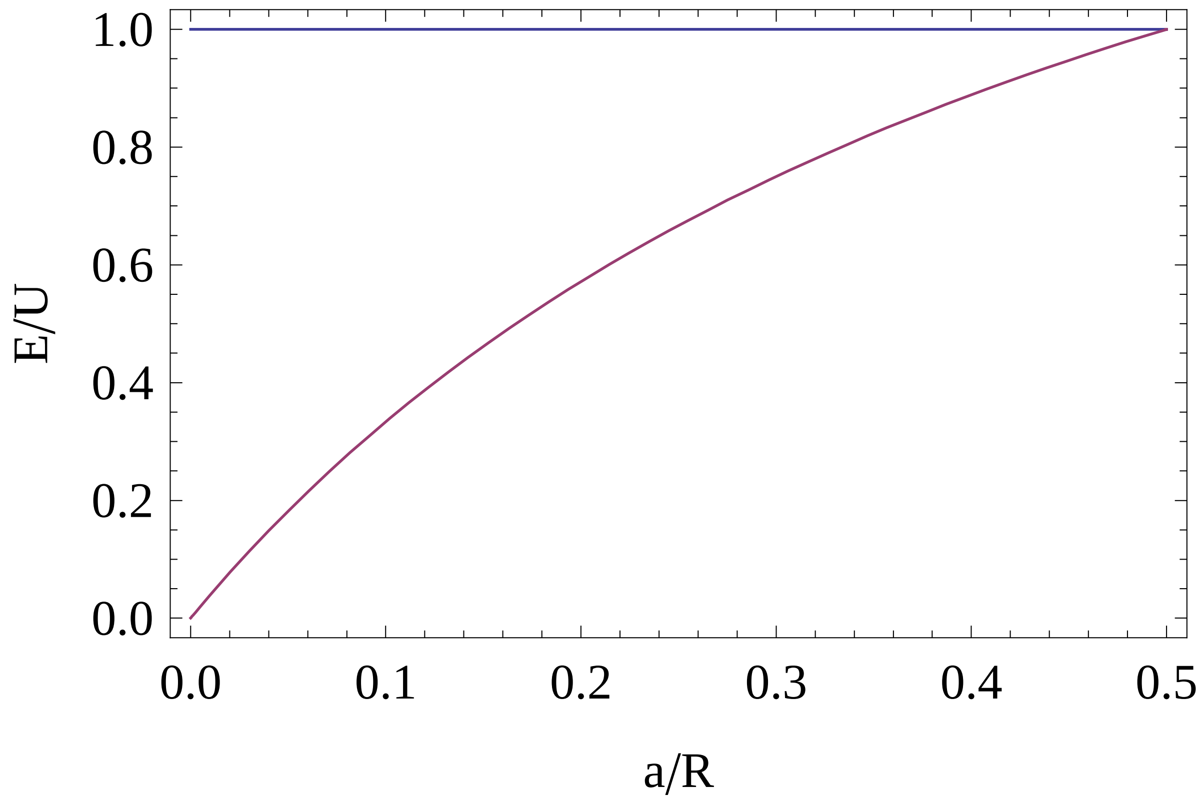


Figure 6: Plotted is the ratio of the exact interaction energy of a weakly-coupled sphere above a Dirichlet plane to the PFA.

II. Exact Results–Weak Coupling

In weak coupling it is possible to derive the exact (scalar) interaction between two potentials

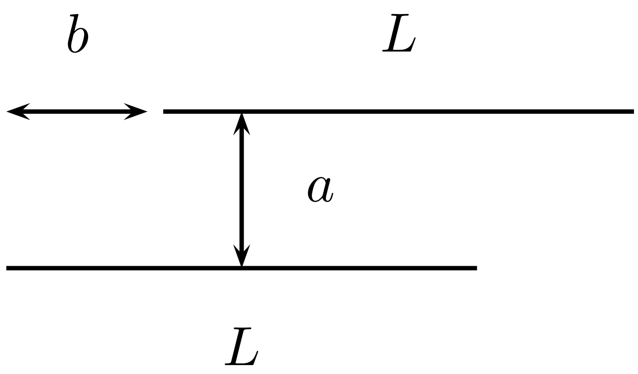
$$2D: \qquad \frac{E}{L_z} = -\frac{1}{32\pi^3}\int (d\mathbf{r}_\perp)(d\mathbf{r}'_\perp)\frac{V_1(\mathbf{r}_\perp)V_2(\mathbf{r}'_\perp)}{|\mathbf{r}-\mathbf{r}'|^2},$$

$$3D: \qquad E = -\frac{1}{64\pi^3}\int (d\mathbf{r})(d\mathbf{r}')\frac{V_1(\mathbf{r})V_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}.$$

Exact Results for Finite Plates

Consider two plates of finite length L, offset by an amount b, separated by a distance a:

$$V_1(\mathbf{r}_\perp) = \lambda_1 \delta(y)\theta(x)\theta(L-x),$$
$$V_2(\mathbf{r}'_\perp) = \lambda_2 \delta(y'-a)\theta(x'-b)\theta(L+b-x'),$$



Exact Results for Finite Plates (cont

This gives an explicit result for the energy between the plate

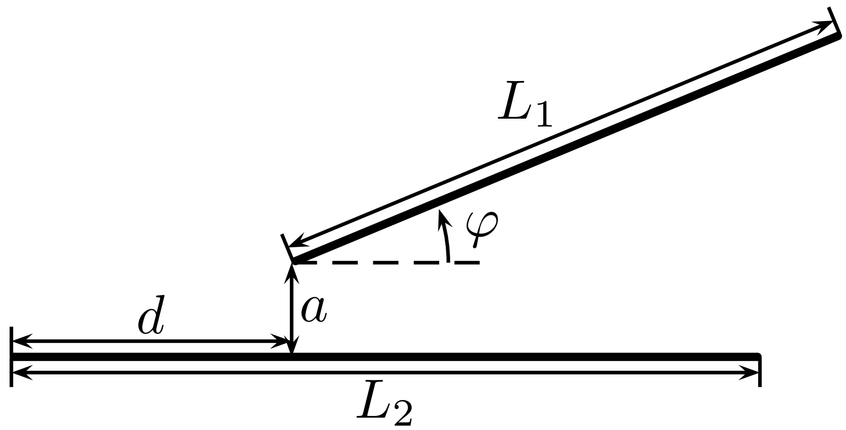
$$\frac{E}{L_z} = -\frac{\lambda_1\lambda_2}{32\pi^3}\left[-2g(b/a) + g((L-b)/a) + g((L+b)/a)\right]$$

where

$$g(x) = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) = -\mathrm{Re}(1+ix)\ln(1+ix).$$

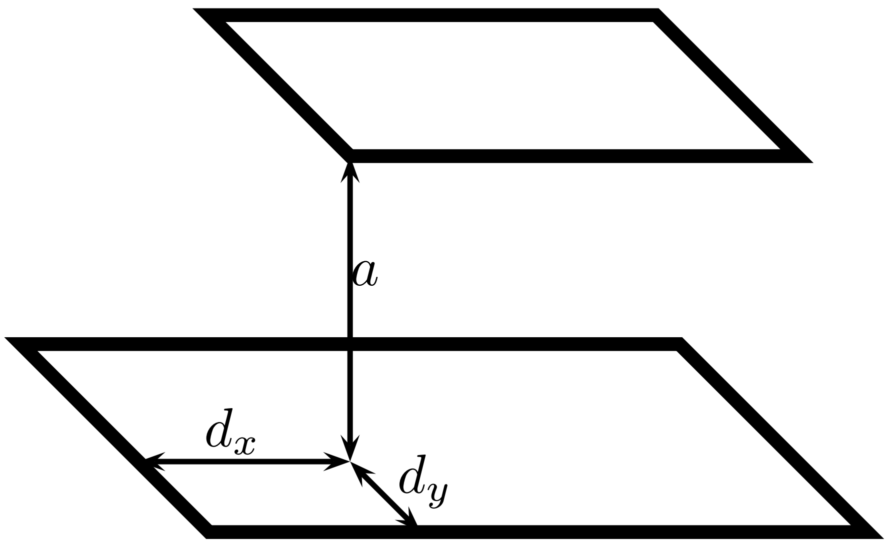
We can consider arbitrary lengths and orientations, in 3 dimensions, for the plates. [J. Wagner et al.]

Tilted plates



Explicit interaction energy can be given in terms of Ti_2, inverse tangent integral. For fixed CM, for $L_1 \to L$, $L_2 \to \infty$, $d \to -\infty$, and $L > 2a$, equilibrium position is at $\phi = \pi/2$.

Rectangular Parallel Plates



As $a \to 0$,

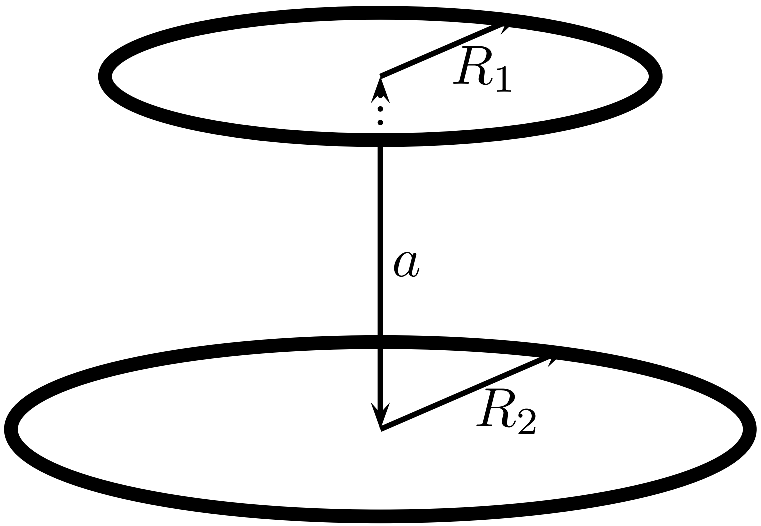
$$\frac{F}{A} = -\frac{\lambda_1\lambda_2}{32\pi^2 a^2}(1 + c_1 a + c_2 a^2 + \dots)$$

Correction to Lifshitz formula

- If upper plate is completely above lower plate, $c_1 = 0$.
- If plates are of the same size and aligned,

$$c_1 = -\frac{1}{\pi}\frac{\text{Perimeter}}{\text{Area}}.$$

Coaxial disks

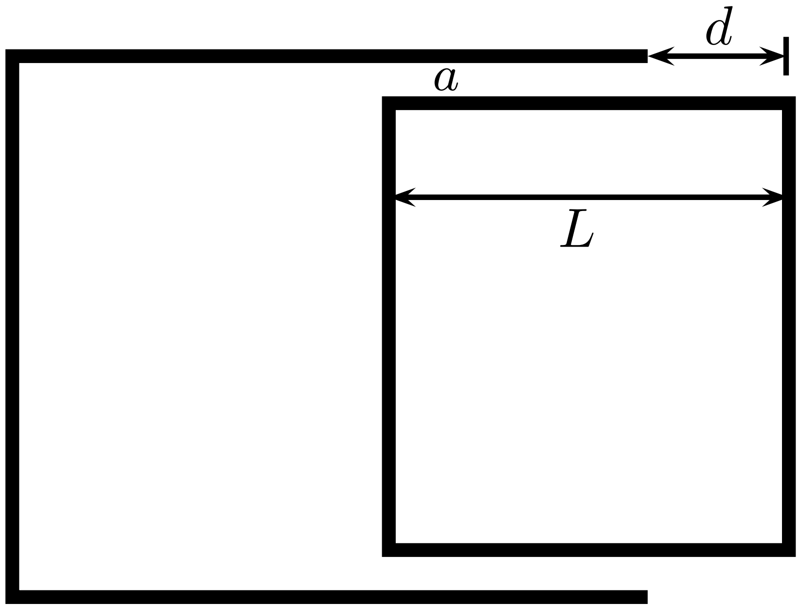


- If $R_1 < R_2$, $c_1 = 0$.
- If $R_1 = R_2$, $c_1 = -\frac{1}{\pi}\frac{\text{Perimeter}}{\text{Area}}$.

Salient Features–two thin plates

- Two plates of finite length experience a lateral force so that they wish to align in the position of maximum symmetry.
- In this symmetric configuration, there is a torque about the CM of a single plate so that it tends to seek perpendicular orientation with respect to the other plate.
- First correction to Lifshitz formula is geometrical.

Relevance to Casimir Pistol



III. Summing van der Waals forces

The (retarded dispersion) van der Waals potential between polarizable molecules is given by

$$V = -\frac{23}{4\pi}\frac{\alpha_1\alpha_2}{r^7}, \quad \alpha = \frac{\varepsilon - 1}{4\pi N}.$$

This allows us to consider in the same vein (electromagnetic) interaction between distinct dilute dielectric bodies of arbitrary shape.

Derivation of vdW interaction

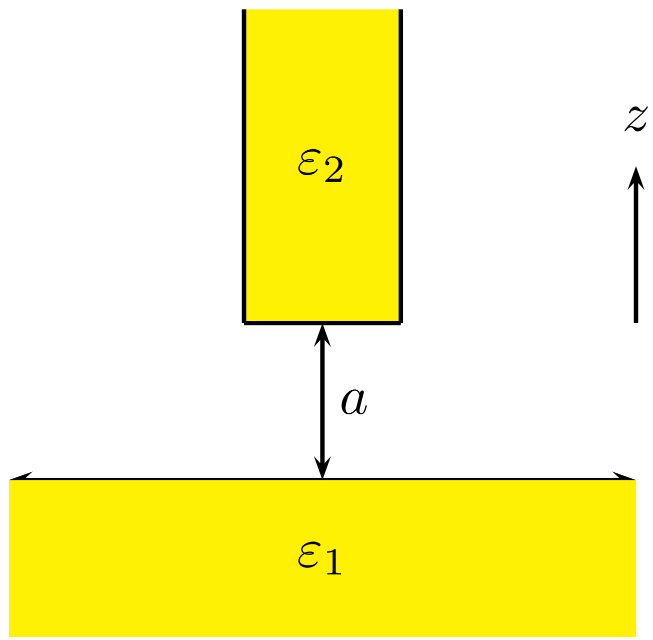
This vdW potential may be directly derived from

$$W = -\frac{i}{2}\mathrm{Tr}\,\ln\frac{\Gamma}{\Gamma_0} \approx -\frac{i}{2}\mathrm{Tr}\,V_1\Gamma_0 V_2\Gamma_0,$$

where

$$\Gamma_0 = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{1}\frac{e^{-|\zeta||\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \mathbf{1}.$$

Force between slab/infinite plate



—No correction to Lifshitz formula

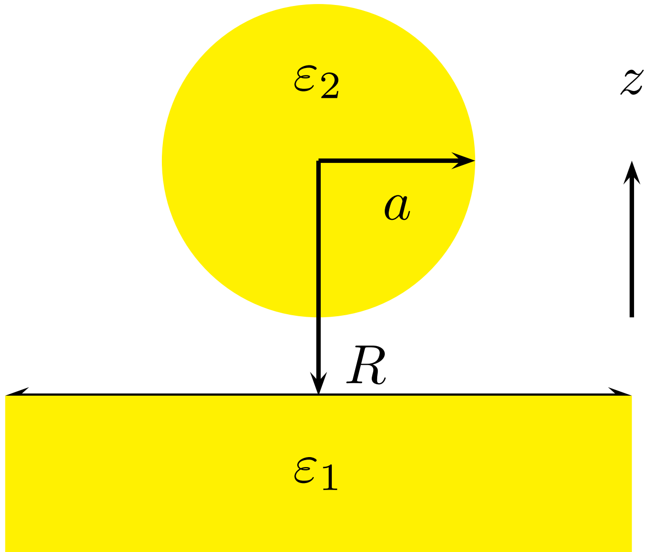
If the cross sectional area of the finite slab is A, the force between the slabs is

$$\frac{F}{A} = -\frac{23}{640\pi^2}\frac{1}{a^4}(\varepsilon_1 - 1)(\varepsilon_2 - 1),$$

the Lifshitz formula for infinite (dilute) slabs.

Note that there is no correction due to the finite area of the slab.

Force between sphere and plate



$$E = -\frac{23}{640\pi^2}(\varepsilon_1 - 1)(\varepsilon_2 - 1)\frac{4\pi a^3/3}{R^4}\frac{1}{(1 - a^2/R^2)^2},$$

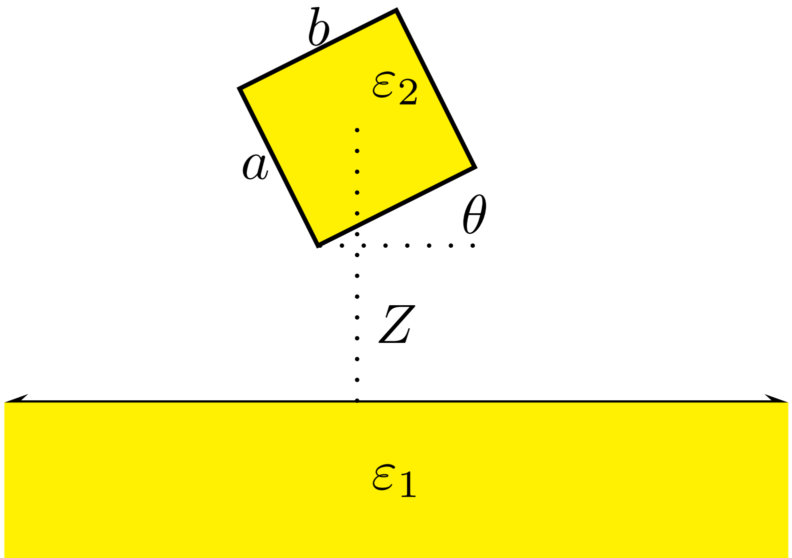
— comparison with PFA

which agrees with the PFA in the short separation limit, $R - a = \delta \ll a$:

$$F_{\rm PFA} = 2\pi a \mathcal{E}_{\|}(\delta) = -\frac{23}{640\pi^2}(\varepsilon_1 - 1)(\varepsilon_2 - 1)\frac{2\pi a}{3\delta^3},$$

with an exact correction, intermediate between that for scalar 1/2(Dirichlet+Neumann) and electromagnetic perfectly-conducting boundaries.

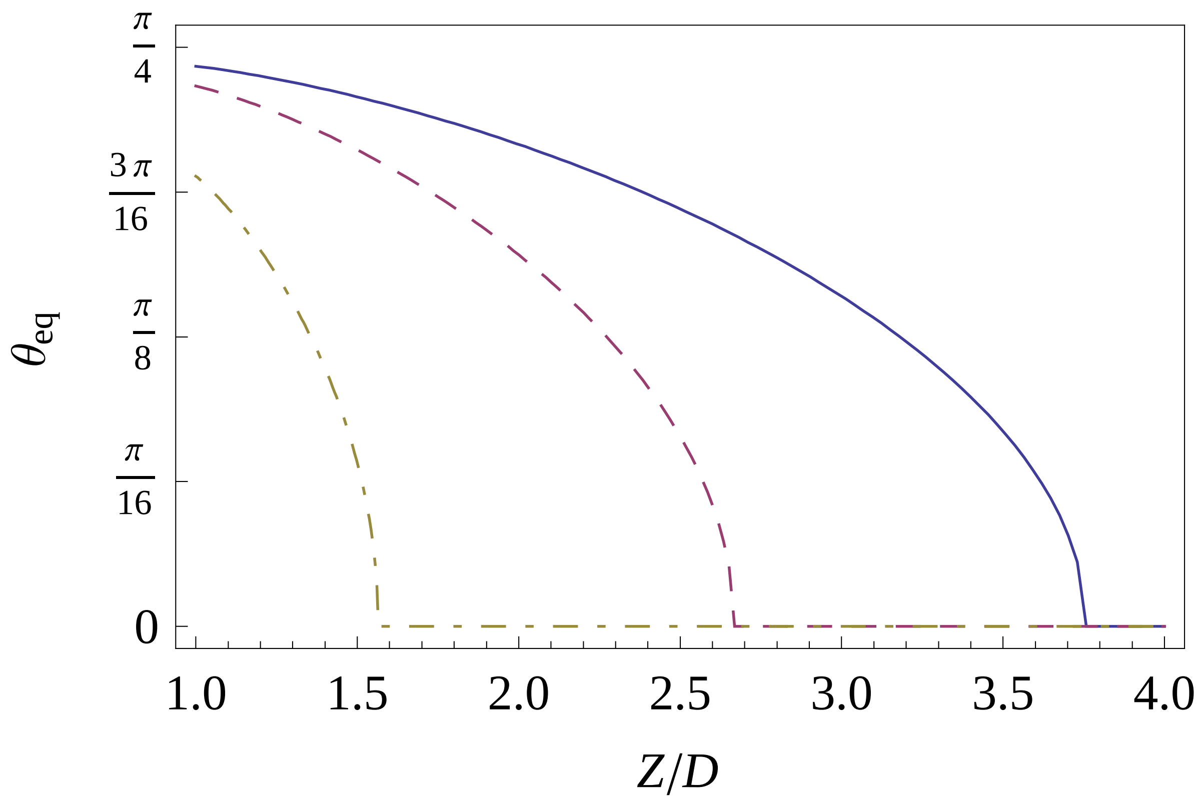
Energy between slab and plate



Torque between slab and plate

Generically, the shorter side wants to align with the plate, which is obvious geometrically, since that (for fixed center of mass position) minimizes the energy. However, if the slab has square cross section, the equilibrium position occurs when a corner is closest to the plate, also obvious geometrically. But if the two sides are close enough in length, a nontrivial equilibrium position between these extremes can occur.

Nontrivial equilibria



Stable equilibria

The stable equilibrium angle of a slab above an infinite plate for given b/a ratios 0.95, 0.9, and 0.7, respectively given by solid, dashed, and dot-dashed lines. For large enough separation, the shorter side wants to face the plate, but for

$$Z < Z_0 = \frac{a}{2}\sqrt{\frac{2a^2 + 5b^2 + \sqrt{9a^4 + 20a^2b^2 + 20b^4}}{5\left(a^2 - b^2\right)}}$$

the equilibrium angle increases, until finally at $Z = D = \sqrt{a^2 + b^2}/2$ the slab touches the plate at an angle $\theta = \arctan b/a$.

Interaction between ∥ cylinders

$$\frac{E}{L} = -\frac{23}{60\pi}(\varepsilon_1 - 1)(\varepsilon_2 - 1)\frac{a^2 b^2}{R^6}$$
$$\times \frac{1 - \frac{1}{2}\left(\frac{a^2+b^2}{R^2}\right) - \frac{1}{2}\left(\frac{a^2-b^2}{R^2}\right)^2}{\left[\left(1 - \left(\frac{a+b}{R}\right)^2\right)\left(1 - \left(\frac{a-b}{R}\right)^2\right)\right]^{5/2}}.$$

Interaction between spheres

$$E = -\frac{23}{1920\pi}\frac{(\varepsilon_1 - 1)(\varepsilon_2 - 1)}{R}\left\{ \ln\left(\frac{1-\left(\frac{a-b}{R}\right)^2}{1-\left(\frac{a+b}{R}\right)^2}\right) + \frac{4ab}{R^2}\frac{\frac{a^6-a^4b^2-a^2b^4+b^6}{R^6} - \frac{3a^4-14a^2b^2+3b^4}{R^4} + 3\frac{a^2+b^2}{R^2} - 1}{\left[\left(1-\left(\frac{a-b}{R}\right)^2\right)\left(1-\left(\frac{a+b}{R}\right)^2\right)\right]^2}\right\}$$

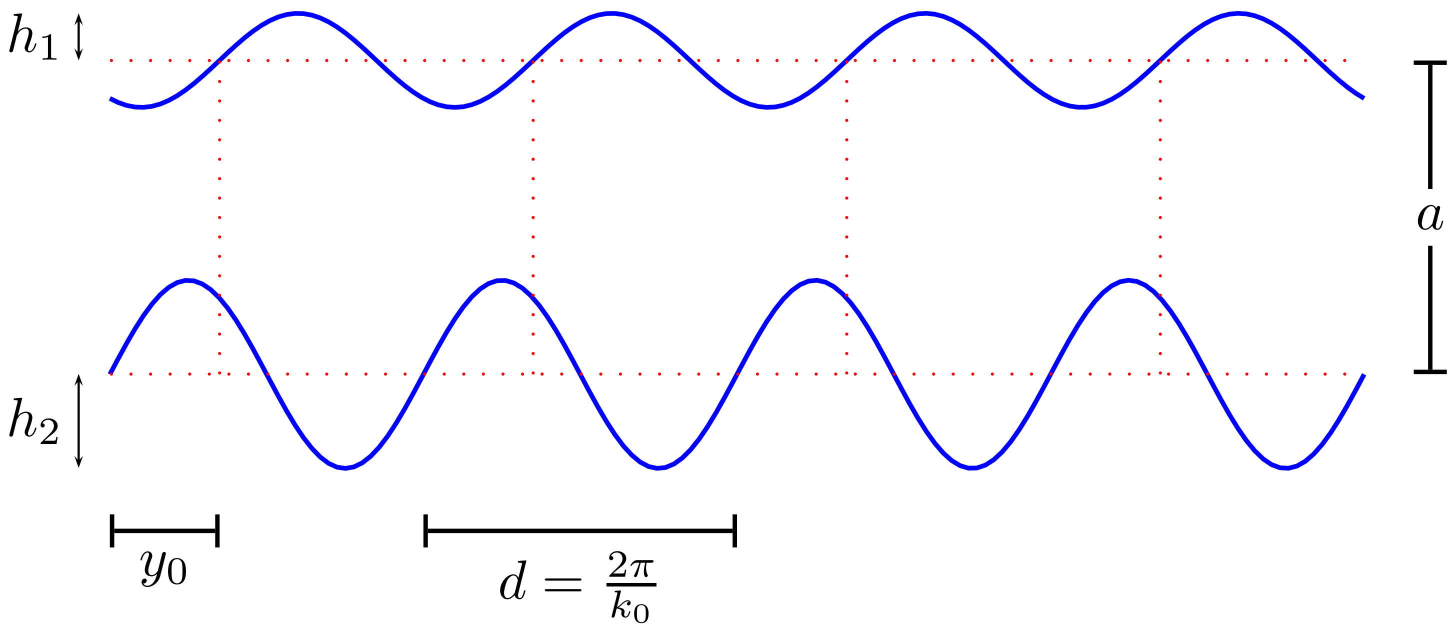
PFA and sphere-plate

This expression, which is rather ugly, may be verified to yield the proximity force theorem:

$$E \to U = -\frac{23}{640\pi}\frac{a(R-a)}{R\delta^2}, \qquad \delta = R - a - b \ll a, b.$$

It also, in the limit $b \to \infty$, $R \to \infty$ with $R - b = Z$ held fixed, reduces to the result for the interaction of a sphere with an infinite plate.

IV. Noncontact gears

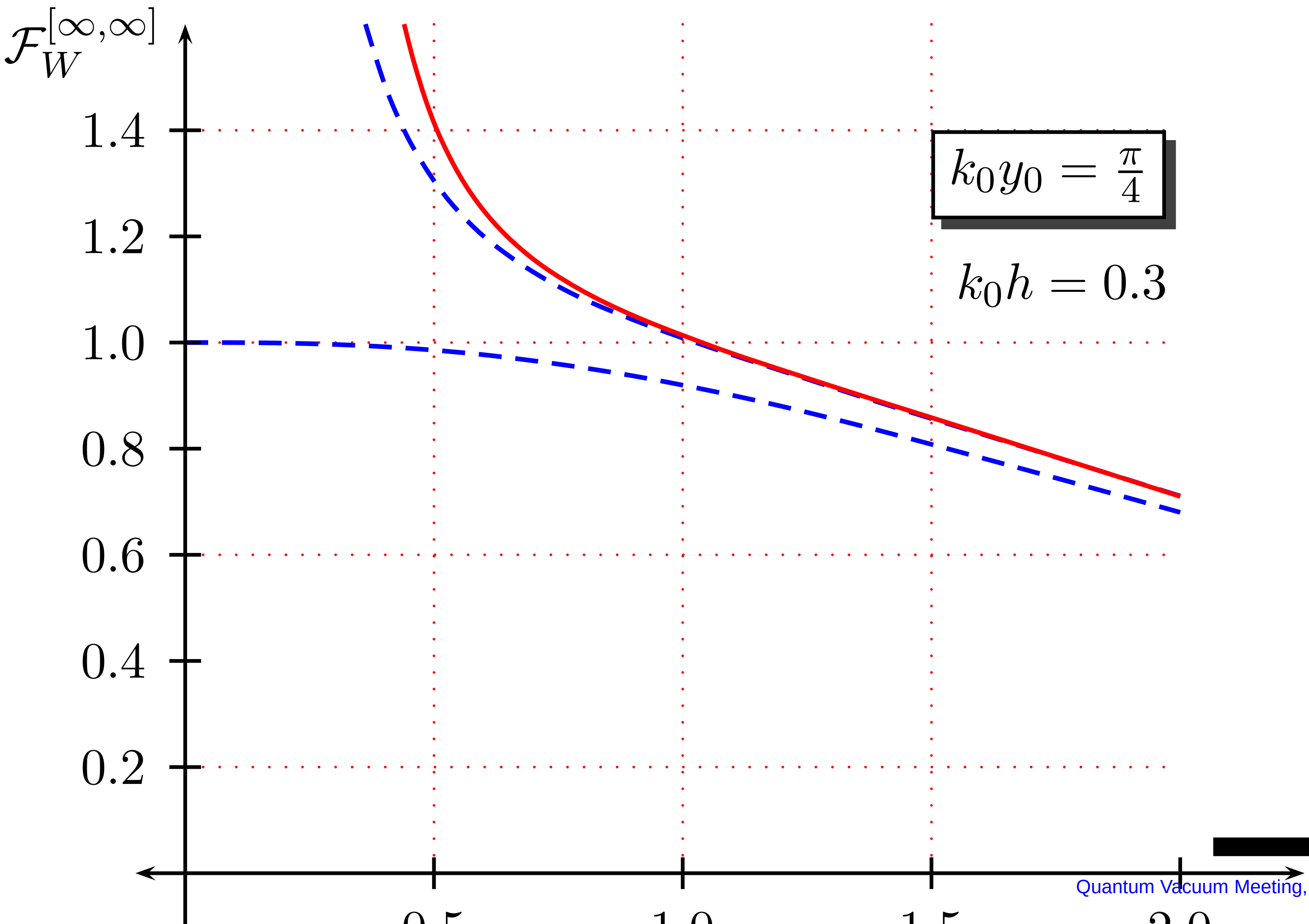


Pertubation theory

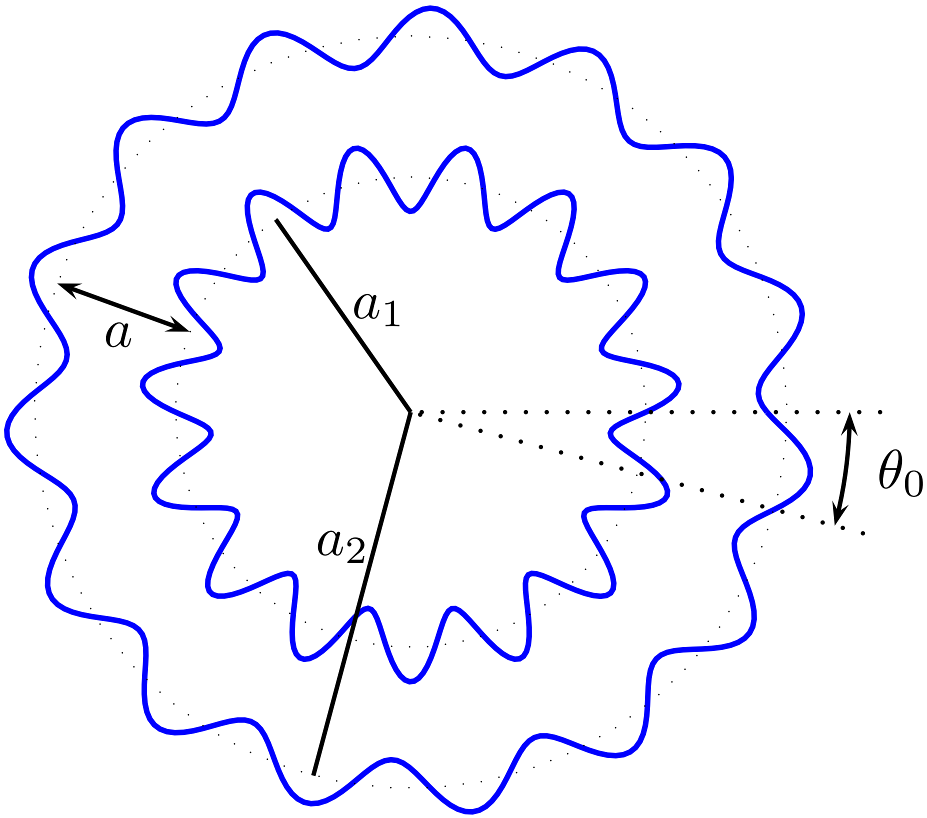
Here we compute the *lateral* force between the offset corrugated plates. The Dirichlet and electromagnetic cases were previously considered by Kardar and Emig, to second order in corrugation amplitudes. We have carried out the calculations to fourth order. In weak coupling we can calculate to all orders, and verify that fourth order is very accurate, provided $k_0 h \ll 1$.

$$\mathcal{F} = \frac{F_{\mathrm{Lat}}}{|F^{(0)}_{\mathrm{Cas}}|(h_1 h_2/a^2) k_0 a \sin(k_0 y)}$$

Weak coupling limit



Concentric corrugated cylinders



Casimir torque per unit area

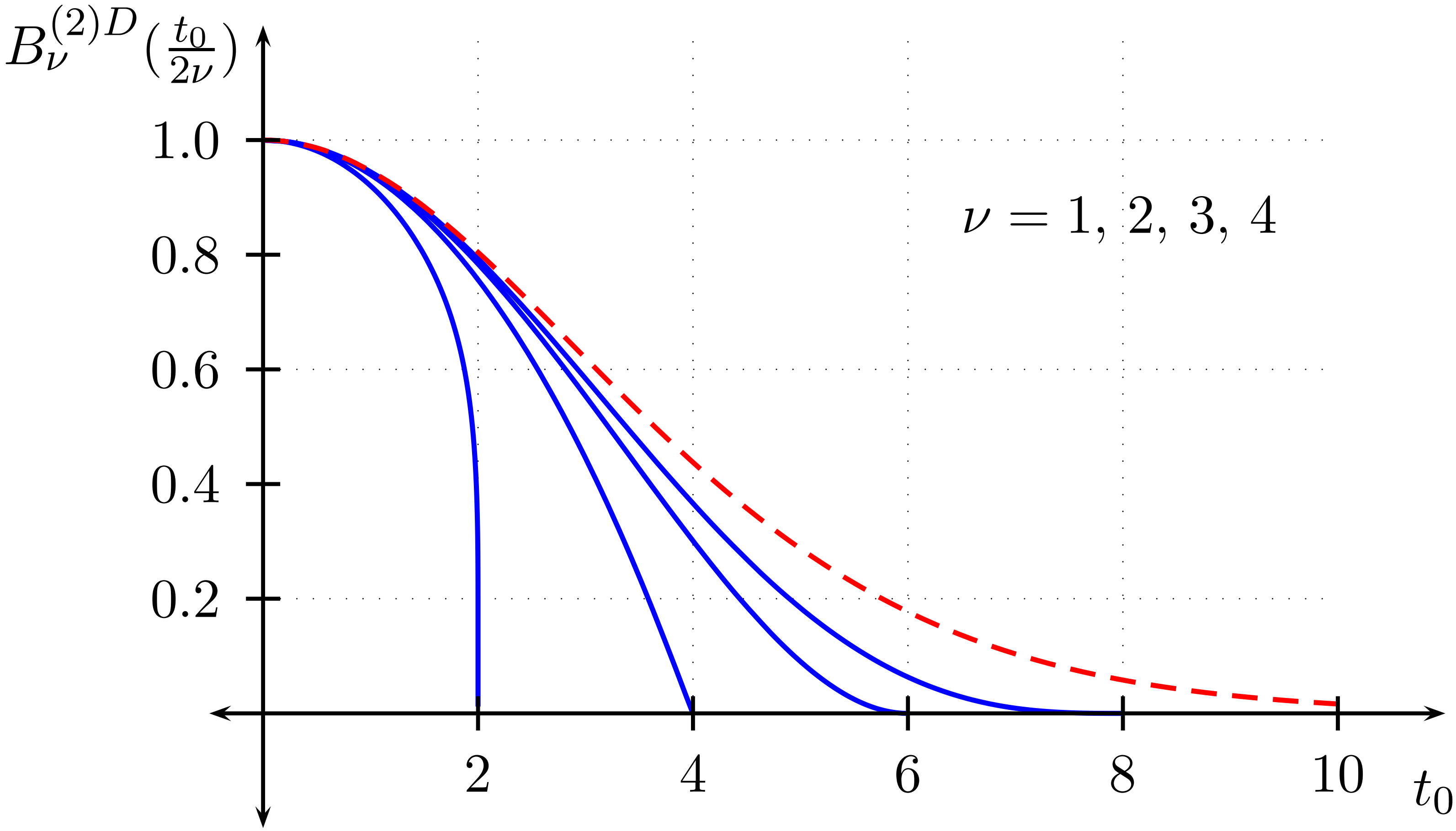
For corrugations given by δ-function potentials with sinusoidal amplitudes:

$$h_1(\theta) = h_1 \sin \nu(\theta + \theta_0),$$
$$h_2(\theta) = h_2 \sin \nu\theta$$

the torque to lowest order in the corrugations in strong coupling (Dirichlet limit) ($\alpha = (a_2 - a_1)/(a_2 + a_1)$)

$$\frac{\tau^{(2)D}}{2\pi R L_z} = \nu \sin \nu\theta_0 \frac{\pi^2}{240 a^3} \frac{h_1}{a} \frac{h_2}{a} B_\nu^{(2)D}(\alpha).$$

Dirichlet limit of cylindrical gears



A similar result can be found for weak coupling

V. Comments and Prognosis

- The methods proposed are in fact not particularly novel, and illustrate the ability of physicists to continually rediscover old methods.
- What is new is the recognition that one can evaluate continuum determinants (or infinitely dimensional discrete ones) accurately numerically, and in some cases even exactly in closed form.
- This is making it possible to compute Casimir forces for geometries previously inaccessible.

New results

- It is indeed remarkable, if perhaps not surprising in retrospect, to see that closed form expressions can be obtained for the interaction between spheres and between parallel cylinders in weak coupling.
- These results demonstrate most conclusively the unreliability of the proximity force approximation (of course, the proximity force theorem holds true).
- This methodology has been used to obtain new results for non-contact gears. (Two papers appearing momentarily in PRD.)