[cylinder kernel of $\mathbf{R}^2$ in polar coordinates]
We have:

$$\begin{aligned}
2\pi T(t,\mathbf{r},\mathbf{r}') &= \frac{t}{(t^2+|\mathbf{r}-\mathbf{r}'|^2)^{\frac{3}{2}}} && (1)\\
&= \frac{t}{(t^2+(r\cos\theta-r'\cos\theta')^2+(r\sin\theta-r'\sin\theta')^2)^{\frac{3}{2}}} && (2)\\
&= \frac{t}{(t^2+r^2+r'^2-2rr'\cos(\theta-\theta'))^{\frac{3}{2}}} && (3)\\
&= 2\pi\frac{\partial\bar{T}}{\partial t} && (4)
\end{aligned}$$

where $2\pi\bar{T} = ((t^2+r^2+r'^2-2rr'\cos(\theta-\theta'))^{-\frac{1}{2}}$
Consider:

$$\frac{\partial^2 T}{\partial t^2}+\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}=0 \tag{5}$$

We assume $T(\theta+\theta_1)=T(\theta)$ and

$$T(0,\mathbf{r},\mathbf{r}')=\frac{1}{r}\delta(r-r')\delta(\theta-\theta') \tag{6}$$

Expand T in Fourier sum in θ:

$$\begin{aligned}
T(t,r,\theta) &= \sum_{n=-\infty}^{\infty} e^{in\theta(\frac{2\pi}{\theta_1})}T_n(t,r) && (7)\\
T_n(t,r) &= \frac{1}{\theta_1}\int_0^{\theta_1} e^{-in\theta(\frac{2\pi}{\theta_1})}T(t,r,\theta) && (8)
\end{aligned}$$

Hence

$$\frac{\partial^2 T_n}{\partial t^2}+\frac{\partial^2 T_n}{\partial r^2}+\frac{1}{r}\frac{\partial T_n}{\partial r}-\frac{n^2}{r^2}\left(\frac{2\pi}{\theta_1}\right)^2 T_n=0 \tag{9}$$

From (6) and (8), take $\theta'=0$ WLOG, we obtain

$$\begin{aligned}
T_n(0,r) &= \frac{1}{\theta_1}\int_0^{\theta_1} d\theta e^{-in\theta(\frac{2\pi}{\theta_1})}\frac{1}{r}\delta(r-r')\delta(\theta-\theta') && (10)\\
&= \frac{1}{\theta_1}e^{-in\theta'(\frac{2\pi}{\theta_1})}\frac{1}{r}\delta(r-r') && (11)
\end{aligned}$$

Try $T_{\text{sep}}(t,r)=T(t)R(r)$. Let $\lambda=\frac{2n\pi}{\theta_1}$. Then

$$\begin{aligned}
T''R+TR''+\frac{1}{r}TR'-\frac{\lambda^2}{r^2}TR=0 && (12)\\
\frac{T''}{T}+\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}-\frac{\lambda^2}{r^2}=0 && (13)
\end{aligned}$$

Let

$$-\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{\lambda^2}{r^2} \tag{14}$$
$$= -\omega^2 \tag{15}$$

So that

$$T = e^{-\omega t} \tag{16}$$
$$R'' + \frac{1}{r}R' + \left(\omega^2 R - \frac{\lambda^2}{r^2}\right) R = 0 \tag{17}$$

The solution of (17) has the form of Bessel functions $J_{|\lambda|}(\omega r)$ (and not $Y_{|\lambda|}(\omega r)$ because of the minimal irregularity at $r = 0$). Now we have

$$T_n(t,r) = \int_0^\infty \omega d\omega \tilde{T}(\omega) J_{|\lambda|}(\omega r) e^{-\omega t} \tag{18}$$

When $t = 0$, from (11), we have

$$T_n(0,r) = \frac{e^{-i\lambda\theta'}\delta(r - r')}{\theta_1 r} = P(r) \tag{19}$$

and from (18):

$$T_n(0,r) = \int_0^\infty \omega d\omega \tilde{T}(\omega) J_{|\lambda|}(\omega r) \tag{20}$$

So one can solve for $\tilde{T}(\omega)$:

$$\tilde{T}(\omega) = \int_0^\infty r dr J_{|\lambda|}(\omega r) P(r) \tag{21}$$
$$= \frac{1}{\theta_1}\int_0^\infty dr J_{|\lambda|}(\omega r) e^{-i\lambda\theta'}\delta(r - r') \tag{22}$$
$$= \frac{e^{-i\lambda\theta'}}{\theta_1} J_{|\lambda|}(\omega r') \tag{23}$$

From this one can get

$$T(t,r,\theta) = \int_0^\infty \omega d\omega \sum_{n=-\infty}^{\infty} \tilde{T}_n(\omega) J_{|\lambda|}(\omega r) e^{in\theta} e^{-\omega t} \tag{24}$$
$$T(t,r,\theta,r',\theta') = \frac{1}{\theta_1}\int_0^\infty \omega d\omega \sum_{n=-\infty}^{\infty} e^{i\lambda(\theta-\theta')} e^{-\omega t} J_{|\lambda|}(\omega r) J_{|\lambda|}(\omega r') \tag{25}$$

The formula for $\bar{T}$ is the same with $\omega d\omega$ replaced by $-d\omega$. From Gradshteyn–Ryzhik, (6.612.3), we get a Legendre function of the second kind:

$$T(t, r, \theta, r', \theta') = \frac{1}{\theta_1} \sum_{n=-\infty}^{\infty} e^{i\lambda(\theta-\theta')} \frac{1}{\sqrt{rr'}} Q_{|\lambda|-\frac{1}{2}}(\cosh u_0) \tag{26}$$

where $\cosh u_0 \equiv \frac{t^2+r^2+r'^2}{2rr'}$. If we follow equation (2.17) and (2.18) of Smith, we'll get the same final answer as tft3d.