[cylinder kernel of $\mathbf{R}^3$ in polar coordinates]
We have:

$$\pi^2 T(t,\mathbf{r},\mathbf{r}') = \frac{t}{(t^2+|\mathbf{r}-\mathbf{r}'|^2)^2} \tag{1}$$

$$= \frac{t}{(t^2+r^2+r'^2-2rr'\cos(\theta-\theta')+(z-z')^2)^2} \tag{2}$$

$$= \pi^2 \frac{\partial \bar{T}}{\partial t} \tag{3}$$

where

$$-2\pi^2 \bar{T} = \frac{1}{t^2+r^2+r'^2-2rr'\cos(\theta-\theta')+(z-z')^2} \tag{4}$$

Consider:

$$\frac{\partial^2 T}{\partial t^2} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{5}$$

We assume $T(\theta+\theta_1) = T(\theta)$ and

$$T(0,\mathbf{r},\mathbf{r}') = \frac{1}{r}\delta(r-r')\delta(\theta-\theta')\delta(z-z') \tag{6}$$

Expand T in Fourier sum in θ:

$$T(t,r,\theta,z) = \sum_{n=-\infty}^{\infty} e^{in\theta(\frac{2\pi}{\theta_1})} T_n(t,r,z) \tag{7}$$

$$T_n(t,r,z) = \frac{1}{\theta_1}\int_0^{\theta_1} e^{-in\theta(\frac{2\pi}{\theta_1})} T(t,r,\theta,z) \tag{8}$$

Hence

$$\frac{\partial^2 T_n}{\partial t^2} + \frac{\partial^2 T_n}{\partial r^2} + \frac{1}{r}\frac{\partial T_n}{\partial r} - \frac{n^2}{r^2}\left(\frac{2\pi}{\theta_1}\right)^2 T_n = 0 \tag{9}$$

From (6) and (8), take $\theta' = 0$ WLOG, we obtain

$$T_n(0,r,z) = \frac{1}{\theta_1}\int_0^{\theta_1} d\theta e^{-in\theta(\frac{2\pi}{\theta_1})} \frac{1}{r}\delta(r-r')\delta(\theta-\theta')\delta(z-z') \tag{10}$$

$$= \frac{1}{\theta_1} e^{-in\theta'(\frac{2\pi}{\theta_1})} \frac{1}{r}\delta(r-r')\delta(z-z') \tag{11}$$

Try $T_{\text{sep}}(t,r) = T(t)R(r)Z(z)$. Let $\lambda = \frac{2n\pi}{\theta_1}$. Then

$$T''RZ + TR''Z + TRZ'' + \frac{1}{r}TR'Z - \frac{\lambda^2}{r^2}TR = 0 \tag{12}$$

$$\frac{T''}{T} + \frac{R''}{R} + \frac{Z''}{Z} + \frac{1}{r}\frac{R'}{R} - \frac{\lambda^2}{r^2} = 0 \tag{13}$$

Let

$$-\frac{T''}{T}-\frac{Z''}{Z} = \frac{R''}{R}+\frac{1}{r}\frac{R'}{R}-\frac{\lambda^2}{r^2} \tag{14}$$
$$= -\omega^2 \tag{15}$$

So that

$$T = e^{-\omega' t} \tag{16}$$
$$Z = e^{ikz} \tag{17}$$
$$R''+\frac{1}{r}R'+((\omega'^2+k^2)R-\frac{\lambda^2}{r^2})R=0 \tag{18}$$

This implies $T''=(\omega^2+k^2)T$. Let $\omega'^2=\omega^2+k^2$. The solution of (18) has the form of Bessel functions $J_{|\lambda|}(\omega r)$ (and not $Y_{|\lambda|}(\omega r)$ because of the minimal irregularity at $r=0$). Now we have

$$T_n(t,r,z)=\int_0^\infty \omega d\omega \int_{-\infty}^\infty dk\tilde{T}(\omega,k)J_{|\lambda|}(\omega r)e^{-\omega' t}e^{ikz} \tag{19}$$

When $t=0$, from (11), we have

$$T_n(0,r,z)=\frac{e^{-i\lambda\theta'}\delta(r-r')\delta(z-z')}{\theta_1 r}=P(r,z) \tag{20}$$

and from (19):

$$T_n(0,r,z)=\int_0^\infty \omega d\omega \int_{-\infty}^\infty dk\tilde{T}(\omega,k)J_{|\lambda|}(\omega r)e^{ikz} \tag{21}$$

So one can solve for $\tilde{T}(\omega,k)$:

$$\tilde{T}(\omega,k) = \frac{1}{2\pi}\int_{-\infty}^\infty dz\int_0^\infty rdrJ_{|\lambda|}(\omega r)P(r,z)e^{-ikz} \tag{22}$$
$$= \frac{1}{2\pi\theta_1}\int_0^\infty drJ_{|\lambda|}(\omega r)e^{-i\lambda\theta'-ikz'}\delta(r-r') \tag{23}$$
$$= \frac{e^{-i\lambda\theta'}}{2\pi\theta_1}J_{|\lambda|}(\omega r')e^{-ikz'} \tag{24}$$

From this one can get

$$T(t,r,\theta,z) = \int_0^\infty \omega d\omega\int_{-\infty}^\infty dk\sum_{n=-\infty}^\infty \tilde{T}_n(\omega)J_{|\lambda|}(\omega r)e^{in\theta}e^{-\omega' t}e^{ikz} \tag{25}$$
$$T(t,r,\theta,z,r',\theta',z') = \frac{1}{2\pi\theta_1}\sum_{n=-\infty}^\infty\int_0^\infty \omega d\omega\int_{-\infty}^\infty dkJ_{|\lambda|}(\omega r)J_{|\lambda|}(\omega r') \times e^{i\lambda(\theta-\theta')}e^{-\omega' t}e^{ik(z-z')} \tag{26}$$

The formula for $\bar{T}$ is the same with $\omega d\omega$ replaced by $-\frac{d\omega}{\omega'} = -d\omega(\omega^2+k^2)^{-1/2}$. So

$$T(t,r,\theta,z,r',\theta',z') = \frac{1}{2\pi\theta_1}\sum_{n=-\infty}^{\infty} e^{i\lambda(\theta-\theta')}\int_0^\infty \omega d\omega J_{|\lambda|}(\omega r)J_{|\lambda|}(\omega r') \times \int_{-\infty}^{\infty} dk e^{-(\omega^2+k^2)^{\frac{1}{2}}t} e^{ik(z-z')} \tag{27}$$

If we use (3.961.2) of Gradshteyn–Rhyzhik, $\int_0^\infty \exp[-\beta\sqrt{\gamma^2+x^2}]\cos ax \frac{dx}{\sqrt{\gamma^2+x^2}} = K_0(\gamma\sqrt{a^2+\beta^2})$, then the k integral becomes $2K_0(\omega\zeta)$, where $\zeta = \sqrt{t^2+(z-z')^2}$. Hence we have

$$\bar{T}(t,r,\theta,z,r',\theta',z') = -\frac{1}{\pi\theta_1}\sum_{n=-\infty}^{\infty} e^{i\lambda(\theta-\theta')}\int_0^\infty \omega d\omega J_{|\lambda|}(\omega r)J_{|\lambda|}(\omega r')K_0(\omega\zeta) \tag{28}$$

The integral over ω may be found in (6.522.3) of Gradshteyn–Rhyzhik, giving us the Fourier-series representation of $\bar{T}$:

$$\bar{T}(t,r,\theta,z,r',\theta',z') = -\frac{1}{\pi\theta_1}\sum_{n=-\infty}^{\infty} e^{i\lambda(\theta-\theta')}\frac{1}{r_1 r_2}\left(\frac{r_2-r_1}{r_2+r_1}\right)^{|\lambda|} \tag{29}$$

where $r_1 \equiv \sqrt{(r-r')^2+\zeta^2}$ and $r_2 \equiv \sqrt{(r+r')^2+\zeta^2}$.

Let $\frac{r_2-r_1}{r_2+r_1} \equiv e^{-u}$. If we follow the steps illustrated in Smith, we get the final closed form of $\bar{T}$:

$$\bar{T}(t,r,\theta,z,r',\theta',z') = -\frac{1}{2\pi\theta_1 rr'\sinh u}\sum_{n=-\infty}^{\infty} e^{-|\lambda|u+i\lambda(\theta-\theta')} \tag{30}$$

$$= -\frac{1}{2\pi\theta_1 rr'\sinh u}\frac{\sinh(\frac{2\pi}{\theta_1}u)}{\cosh(\frac{2\pi}{\theta_1}u)-\cos\frac{2\pi}{\theta_1}(\theta-\theta')} \tag{31}$$

When $\theta_1 = 2\pi$,

$$\bar{T}(t,r,\theta,z,r',\theta',z') = -\frac{1}{4\pi^2 rr'}\frac{1}{\cosh u-\cos(\theta-\theta')} \tag{32}$$

$$= -\frac{1}{2\pi^2(r^2+r'^2-2rr'\cos(\theta-\theta')+(z-z')^2+(t-t')^2)} \tag{33}$$

which agrees with (4).