Comments on the Cone Calculations

29 June 2008 (revised 10 July)

References are to my "Cosmic strings and wedges" bibiliography of June 7.

1 Agreement of formulas

Mai's formula (31) of `ckpc4d` for $\overline{T}$ agrees, mutatis mutandis, with

- (3.30) of Smith [18] (almost by construction)
- (5) of Dowker [5]
- (21) of Linet [15]
- (3.20) of Lukosz [3] (before taking images)

Lukosz's paper was overlooked by all the later authors. Smith's paper gives the most explicit formulas, obtained by the most elementary methods, which is why Mai has been following it.

Let us recall from Smith [18] (Euclideanized as in `ckpc4d`)

$$u \equiv -\ln \frac{r_2 - r_1}{r_2 + r_1}\,, \tag{1}$$

$$2rr' \cosh u = r^2 + r'^2 + (z-z')^2 + (t-t')^2, \tag{2}$$

$$4rr' \sinh^2 \frac{u}{2} = (r-r')^2 + (z-z')^2 + (t-t')^2. \tag{3}$$

(We usually take $t' = 0$.)

2 Infinite-sheeted case

In (30) of `ckpc4d` we encounter the sum

$$\frac{1}{\theta_1} \sum_{n=-\infty}^{\infty} e^{-2\pi|n|u/\theta_1 + i2\pi n(\theta-\theta')/\theta_1}. \tag{4}$$

In the limit $\theta_1 \to \infty$ it becomes the integral

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dy\, e^{-|y|u + iy(\theta - \theta')} = \frac{1}{\pi} \frac{u}{u^2 + (\theta - \theta')^2} \,. \tag{5}$$

Thus $\overline{T}$ for the infinite-sheeted Sommerfeld–Dowker manifold is

$$\overline{T}(t, r, \theta, z, r', \theta', z') = -\frac{1}{2\pi^2 rr' \sinh u} \frac{u}{u^2 + (\theta - \theta')^2} \,. \tag{6}$$

This agrees, mutatis mutandis, with (7) of Dowker [5].

3 Lower-dimensional case

According to (2.5) Guimarães–Linet [21] (or (11) of Linet [15]) the Euclidean Green function $(-\frac{1}{2}\overline{T})$ in dimension n is obtained from the massive Euclidean Green function in dimension $n-1$ by a Fourier [cosine] transform with respect to the mass. Therefore, the lower-dimensional case can be obtained from the higher one by an inverse transform; and if $m = 0$, the Fourier factor becomes trivial. Thus

$$\overline{T}(t, r, \theta, r'\theta') = 2 \int_0^{\infty} \overline{T}(t, r, \theta, z, r', \theta', 0)\, dz. \tag{7}$$

From (2) with $z' = 0$ and $t' = 0$,

$$2rr' \sinh u\, du = 2z\, dz, \tag{8}$$

and $z = 0$ corresponds to $u = u_0$, defined in (2.16) of Smith [18] and by Mai in `tft3d` by

$$\cosh u_0 = \frac{r^2 + r'^2 + t^2}{2rr'} \,. \tag{9}$$

Therefore, from (31) of `ckpc4d`,

$$\begin{aligned} \overline{T}(t, r, \theta, r'\theta') &= 2rr' \int_{u_0}^{\infty} \overline{T}(t, r, \theta, z, r', \theta', 0) \sinh u \frac{du}{\sqrt{2rr' \cosh u - r^2 - r'^2 - t^2}} \\ &= -\frac{1}{\pi\theta_1} \int_{u_0}^{\infty} \frac{\sinh(\frac{2\pi u}{\theta_1})}{\cosh(\frac{2\pi u}{\theta_1}) - \cos\frac{2\pi(\theta - \theta')}{\theta_1}} \frac{du}{\sqrt{2rr'(\cosh u - \cosh u_0)}}. \end{aligned} \tag{10}$$

This is the same as (17) of `tft3d` after the sum there is evaluated as it was in `ckpc4d`.

In any dimension, $\overline{T}$ can be constructed either

1. as a Poisson kernel for a homogeneous PDE in $t > 0$ with nonhomogeneous data at $t = 0$, by an eigenfunction expansion in the $(r, \theta, z, \ldots)$ variables; or

2. as a Green function for a nonhomogeneous PDE in $-\infty < t < \infty$, by Fourier analysis in $(\theta, z, \ldots, t)$.

My document `ehk`, following [8] and [6], shows how both formulas follow from a full harmonic analysis (of the nonhomogeneous problem) in *all* the variables. Mai has worked on both dimensions (total of 3 and total of 4) by both methods, but concentrated on the two approaches that matched Smith's work, namely, method 1 (ckpc) in dimension 4 and method 2 (tft) in dimension 3.

4 Lower-dimensional infinite-sheeted case

Combining the considerations of the two previous sections, we get for the Sommerfeld–Dowker manifold of total dimension 3 the result

$$\overline{T}(t,r,\theta,r'\theta') = -\frac{1}{\pi^2}\int_{u_0}^{\infty} \frac{u}{u^2+(\theta-\theta')^2}\,\frac{du}{\sqrt{2rr'(\cosh u-\cosh u_0)}}. \tag{11}$$

Note that in the 3D formulas u is simply a variable of integration, but in the 4D formulas it is defined in terms of the coordinates by (1) or (2).