The Wedge Problem

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The basic problem

For definitenesss I will state the equations defining the cylinder kernel, but it will be easy to formulate the analogous wave, heat, quantum (Schrödinger), and resolvent problems. Note that the two-dimensional cylinder kernel is closely related to the three-dimensional resolvent kernel (a.k.a. energy-domain Green function).

Also, I will state the Dirichlet boundary condition, but we should also consider Neumann and mixed (D on one side of wedge, N on the other) conditions.

$$T(t, \mathbf{x}, \mathbf{x}') \quad \text{is defined in } \mathbf{R}^+ \times \Omega \times \Omega.$$

$$\frac{\partial^2 T}{\partial t^2} = -\nabla^2 T \quad \text{for } \mathbf{x} \text{ in } \Omega,$$

$$T(t, \mathbf{x}, \mathbf{x}') = 0 \quad \text{for } \mathbf{x} \text{ in } \partial\Omega,$$

$$T(0, \mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \quad \text{for } \mathbf{x} \text{ in } \Omega,$$

$$T(t, \mathbf{x}, \mathbf{x}') \quad \text{is bounded as } t \to +\infty.$$

Two-dimensional case

$$\Omega = \{(r, \theta) \colon 0 < r < \infty,\ 0 < \theta < \theta_1\},$$

$$ds^2 = dr^2 + r^2\, d\theta^2,$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}\,,$$

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{1}{r}\,\delta(r - r')\delta(\theta - \theta').$$

Three-dimensional case

$$\Omega = \{(r, \theta, z) \colon 0 < r < \infty,\ 0 < \theta < \theta_1\,,\ -\infty < z < \infty\},$$

$$ds^2 = dr^2 + r^2\, d\theta^2 + dz^2,$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\,,$$

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{1}{r}\,\delta(r - r')\delta(\theta - \theta')\delta(z - z'),$$

$$T(t, \mathbf{x}, \mathbf{x}') \quad \text{is bounded as } z \to \pm\infty.$$

The cosmic string (cone) problem

Exactly the same as the foregoing, except that the Dirichlet condition is replaced by the periodicity condition:

$$T_0(t, r, \theta + \theta_1, \mathbf{x}') = T_0(t, r, \theta, \mathbf{x}'), \qquad \frac{\partial T_0}{\partial \theta}(t, r, \theta + \theta_1, \mathbf{x}') = \frac{\partial T_0}{\partial \theta}(t, r, \theta, \mathbf{x}')$$

(and similarly for 3D). When $\theta_1 = 2\pi$, Ω is just $\mathbf{R}^n$ in polar/cylindrical coordinates. (This is loosely called the "free" case.)

Good angles: $\theta_1 = \frac{\pi}{N}$

In other words, the sector considered is one of $2N$ sectors into which Euclidean space is divided. For mixed boundary conditions, θ_1 will be good only if N is even (the number of sectors is a multiple of 4). For the cone, only $\theta_1 = 2\pi$ is good. In that case we know the solution T_0 in Cartesian coordinates, but we'll eventually need its expression in polar coordinates.

Spectral solution

Solve the problem(s) by separation of variables. There may be more than one good way of representing the solution in polar coordinates (see remarks about Lukosz below).

Image (or classical path) solution

Solve the problem(s) in terms of (the free) T_0 at image positions in the $2N$ sectors. Verify equivalence with the spectral solution.

Bad angles

Spectral solution

Solve the Dirichlet and cone problems by separation of variables. Bessel functions of nonintegral order will appear.

Image solutions

Reduce the Dirichlet problem to a cone problem with the cone divided into $2N$ sectors each matching the original wedge; express T in terms of the T_0 of the cone. Here N appears to be an arbitrary positive integer. Verify that the results are equivalent for all integers N.

The third Lukosz paper

Part of the 3D problem has already been done by Lukosz, *Z. Phys.* **262** (1973) 327. (References [15] and [1] are what I call his first and second papers.)

He makes the wedge symmetrical about a Cartesian axis, but I don't see the advantage of doing so (except in the periodic case).

He gets the free T_0 in cylindrical coordinates directly from the Cartesian formula, and relates the resulting T functions to solutions by separation of variables only as an afterthought. I think it should be possible to proceed in the opposite direction.

His reference to "Meixner's edge condition" probably refers to the fact that for large wedge angles, strictly speaking an additional boundary condition at $r = 0$ may be needed to make the operator self-adjoint and the solution unique. The point is that for Bessel's equation of very small order, both Bessel functions may be square-integrable in the neighborhood of 0. For the time being assume that the correct solution is the one that vanishes faster there.