Zeta function regularization of the spectral determinant and vacuum energy of quantum graphs

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Outline

1. Motivation
2. Zeta functions of quantum graphs
3. Spectral determinant
4. Vacuum energy

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Spectrum $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \dots$.

Spectral zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$$

Spectral determinant

$$\prod_{n=1}^{\infty} \lambda_j = \exp\left(-\zeta'(0) \right)$$

Vacuum energy

$$\frac{1}{2} \sum_{n=1}^{\infty} \sqrt{\lambda_j} = \frac{1}{2} \zeta(-1/2)$$

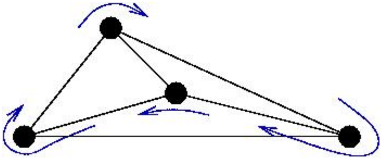
Riemann zeta function

$$\zeta_{\mathrm{R}}(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1} \tag{1}$$

Ihara-Selberg zeta function for a q-regular graph

$$Z(s) = \prod_{[p]} (1 - q^{-|p|s})^{-1} \tag{2}$$

$[p]$ periodic orbits without backtracking or tails.



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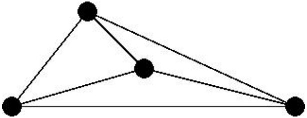
Quantum graphs



Set of vertices $\mathcal{V}$, $V = |\mathcal{V}|$.
Set of bonds $\mathcal{B}$, $B = |\mathcal{B}|$.

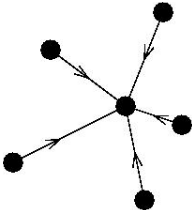
Bond b corresponds to interval $[0, L_b]$.

Hilbert space $\mathcal{H} := \bigoplus_{b=1}^{B} L^2\big([0, L_b]\big)$.

Operator on bond $-\frac{\mathrm{d}^2}{\mathrm{d}x_b^2}$. Define matching conditions at vertices so Laplace op. $-\triangle$ on the graph is self-adjoint.

Example I – Neumann star

- One central vertex with B external vertices (*nodes*).
- Neumann bcs at nodes, $\psi_b'(0) = 0$.
- **Center:** fn continuous, $\psi_b(L_b) = \psi$, and $\sum_b \psi_b'(L_b) = 0$.



Eigenproblem
$-\triangle\psi = k^2\psi$

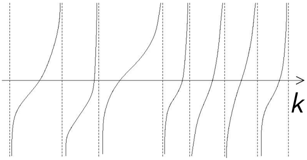
Bcs at the nodes and continuity at the center implies

$$\psi_b(x_b) = \psi \frac{\cos kx_b}{\cos kL_b} \ .$$

Substitute in Neumann condition at the center.

Secular equation

$$\sum_{b=1}^{B} \tan kL_b = 0 \tag{3}$$



- Zeros k_j of secular eqn correspond to evals k_j^2 of Laplace op.
- Poles $\bigcup_{b\in\mathcal{B}}\{(m+1/2)\pi/L_b\}_{m\in\mathbb{Z}}$.
- $\{L_b\}$ incommensurate: Poles and zeros distinct with one zero between every pair of adjacent poles.

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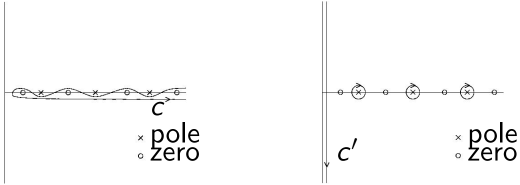
Argument principle

$$\zeta(s) = \sum_{j=1}^{\infty} k_j^{-2s} = \int_c z^{-2s} \frac{f'(z)}{f(z)} \, \mathrm{d}z \tag{4}$$

where $f(z)$ has zeros at $\{k_j\}$.



Provided $|f'(z)/f(z)|$ decays sufficiently transform c to c'.

$$\zeta(s) = \zeta_{\mathrm{Im}}(s) + \zeta_{\mathrm{P}}(s)$$

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$$f(z) = \frac{1}{z}\sum_{b=1}^{B} \tan zL_b$$

Chosen so zeros of secular eqn are zeros of f but $f(0)$ finite.
Integral around c' converges for $0 < \mathrm{Re}\, s < 1$.

Pole contribution

At a pole z_0 of f subtract residue z_0^{-2s}.

$$\begin{aligned}\zeta_{\mathrm{P}}(s) &= \sum_{b=1}^{B}\left(\frac{\pi}{L_b}\right)^{-2s}\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)^{-2s} \\ &= (2^{2s}-1)\zeta_{\mathrm{R}}(2s)\sum_{n}\left(\frac{\pi}{L_b}\right)^{-2s}\end{aligned}$$

Imaginary axis integral

$z = \mathrm{i}t$ and $f(\mathrm{i}t) = -\hat{f}(t)/t$.

$$\hat{f}(t) = \sum_{b=1}^{B} \tanh tL_b \tag{5}$$

$$\begin{aligned}
\zeta_{\mathrm{Im}}(s) &= \frac{1}{2\pi\mathrm{i}} \int_{\infty}^{-\infty} (\mathrm{i}t)^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log\left(f(\mathrm{i}t)\right) \mathrm{d}t \\
&= \frac{\sin \pi s}{\pi} \int_0^{\infty} t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log\left(\frac{1}{t}\hat{f}(t)\right) \mathrm{d}t \qquad 0 < \mathrm{Re}\, s < 1 \\
&= \frac{\sin \pi s}{\pi} \left[\int_0^{1} t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log\left(\frac{1}{t}\hat{f}(t)\right) \mathrm{d}t \right. \\
&\qquad \left. + \int_1^{\infty} t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log \hat{f}(t)\, \mathrm{d}t - \frac{1}{2s}\right] \qquad \mathrm{Re}\, s < 1
\end{aligned}$$

Zeta fn of Neumann star

$$\zeta(s) = \zeta_{\mathrm{Im}}(s) + \zeta_{\mathrm{P}}(s)$$

$$\zeta_{\mathrm{Im}}(s) = \frac{\sin \pi s}{\pi} \left[\int_0^1 t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log \left(\frac{1}{t} \hat{f}(t) \right) \mathrm{d}t + \int_1^\infty t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log \hat{f}(t) \, \mathrm{d}t - \frac{1}{2s} \right] \qquad \mathrm{Re}\, s < 1$$

$$\hat{f}(t) = \sum_{b=1}^{B} \tanh tL_b$$

$$\zeta_{\mathrm{P}}(s) = (2^{2s} - 1)\zeta_{\mathrm{R}}(2s) \sum_{b=1}^{B} \left(\frac{\pi}{L_b} \right)^{-2s}$$

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Equal bond lengths

$L_b = L$ for all $b \in \mathcal{B}$, secular eqn reduces to $\tan kL = 0$.

k-spectrum

$$\left\{ \frac{n\pi}{L} \right\}_{n\in\mathbb{Z}} \quad \text{and} \quad \left\{ \frac{(m+1/2)\pi}{L} \right\}_{m\in\mathbb{Z}} \quad \text{multiplicity } B-1$$

Zeta fn with equal bond lengths

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} \right)^{-2s} + (B-1) \sum_{m=0}^{\infty} \left(\frac{(m+1/2)\pi}{L} \right)^{-2s} \\ &= \left(\frac{\pi}{L} \right)^{-2s} \left((B-1)2^{2s} - B + 2 \right) \zeta_R(2s) \end{aligned}$$

Example II – general star

- B nodes with Dirichlet bcs, $\psi_b(0) = 0$.
- Matching at center defined by two $B \times B$ matrices via

$$\mathbb{A}\boldsymbol{\psi} + \mathbb{B}\boldsymbol{\psi}' = \mathbf{0} \ .$$

Theorem (Kostrykin & Schrader)

$\mathbb{A}, \mathbb{B}$ *define a self-adjoint Laplace op. iff* $\mathrm{rank}(\mathbb{A}, \mathbb{B}) = B$ *and* $\mathbb{A}\mathbb{B}^\dagger = \mathbb{B}\mathbb{A}^\dagger$.

e.g. Neumann matching at center

$$\mathbb{A} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -1 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix} , \qquad \mathbb{B} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \end{pmatrix} .$$

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Secular equation

$\psi_b(x_b) = c_b \sin kx_b$ let $\mathbf{c} = (c_1, \ldots, c_B)^T$ & $\mathbf{L} = \mathrm{diag}\{L_1, \ldots, L_B\}$.
The matching condition at the center is then

$$\Big(\mathbb{A}\sin(k\mathbf{L}) - k\mathbb{B}\cos(k\mathbf{L})\Big)\mathbf{c} = \mathbf{0}\ . \tag{6}$$

k is an eigenvalue iff it is a soln of

$$\det\Big(\mathbb{A}\sin(k\mathbf{L}) - k\mathbb{B}\cos(k\mathbf{L})\Big) = 0\ . \tag{7}$$

Secular equation of a general star

$$\det\begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathrm{I}_B & \frac{1}{k}\tan(k\mathbf{L}) \end{pmatrix} = 0 \tag{8}$$

Zeta function

$$f(z) = \det \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathrm{I}_B & \frac{1}{z}\tan(z\mathbf{L}) \end{pmatrix} \qquad \hat{f}(t) = \det \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathrm{I}_B & \frac{1}{t}\tanh(t\mathbf{L}) \end{pmatrix}$$

Theorem (Zeta fn of general star)

$$\zeta(s) = \frac{\sin \pi s}{\pi}\left[\int_0^1 t^{-2s}\frac{\mathrm{d}}{\mathrm{d}t}\log \hat{f}(t)\,\mathrm{d}t + \int_1^\infty t^{-2s}\frac{\mathrm{d}}{\mathrm{d}t}\log(t^N\hat{f}(t))\,\mathrm{d}t\right]$$
$$- \frac{N\sin \pi s}{2\pi} + (2^{2s}-1)\zeta_R(2s)\sum_{b=1}^{B}\left(\frac{\pi}{L_b}\right)^{-2s} \qquad -\frac{1}{2} < s < 1$$

$$\hat{f}(t) \sim \frac{a_N}{t^N} + \frac{a_{N+1}}{t^{N+1}} + \dots$$

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Example III – general graph

- Matching conditions on whole graph specified by $2B \times 2B$ matrices

$$\mathbb{A}\boldsymbol{\psi} + \mathbb{B}\boldsymbol{\psi}' = \mathbf{0} \ . \tag{9}$$

- Wavefunction on bond $\psi_b(x_b) = c_b \sin kx_b + \hat{c}_b \cos kx_b$.

$$\mathbb{A}\begin{pmatrix} 0 & \mathrm{I} \\ \sin(k\mathbf{L}) & \cos(k\mathbf{L}) \end{pmatrix}\begin{pmatrix} \mathbf{c} \\ \hat{\mathbf{c}} \end{pmatrix} + k\mathbb{B}\begin{pmatrix} \mathrm{I} & 0 \\ -\cos(k\mathbf{L}) & \sin(k\mathbf{L}) \end{pmatrix}\begin{pmatrix} \mathbf{c} \\ \hat{\mathbf{c}} \end{pmatrix} = \mathbf{0}$$

Secular equation

$$\det\left(\mathbb{A} + k\mathbb{B}\begin{pmatrix} -\cot(k\mathbf{L}) & \csc(k\mathbf{L}) \\ \csc(k\mathbf{L}) & -\cot(k\mathbf{L}) \end{pmatrix}\right) = 0$$

Theorem (General graph zeta function)

$$\zeta(s) = \frac{\sin \pi s}{\pi} \left[\int_0^1 t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log \hat{f}(t) \, \mathrm{d}t \int_1^\infty t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log(t^{-N} \hat{f}(t)) \, \mathrm{d}t \right]$$

$$+ \frac{N \sin \pi s}{2\pi s} + \zeta_R(2s) \sum_{b=1}^{B} \left(\frac{\pi}{L_b} \right)^{-2s} \qquad -\frac{1}{2} < s < 1$$

$$\hat{f}(t) = \det \left(\mathbb{A} - t\mathbb{B} \begin{pmatrix} \coth(t\mathbf{L}) & -\mathrm{csch}(t\mathbf{L}) \\ -\mathrm{csch}(t\mathbf{L}) & \coth(t\mathbf{L}) \end{pmatrix} \right)$$

$$\hat{f}(t) \sim \det(\mathbb{A} - t\mathbb{B}) = a_N t^N + \cdots + a_1 t + \det \mathbb{A} \tag{10}$$

Spectral determinant

Formally

$$\det{}'(-\triangle) = \prod_{j=1}^{\infty} \lambda_j$$

From defn of zeta function

$$\zeta'(s) = \sum_{j=1}^{\infty} -(\ln \lambda_j)\, \lambda_j^{-s}$$

Definition (Zeta fn regularization of spectral det)

$$\det{}'(-\triangle) = \exp\Big(-\zeta'(0)\Big)$$

Example I – Neumann star

Zeta fn of Neumann star

$$\zeta(s) = \zeta_{\mathrm{Im}}(s) + \zeta_{\mathrm{P}}(s) \qquad \mathrm{Re}\, s < 1$$

$$\zeta_{\mathrm{Im}}(s) = \frac{\sin \pi s}{\pi} \left[\int_0^1 t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log \left(\frac{1}{t} \hat{f}(t) \right) \mathrm{d}t + \int_1^\infty t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log \hat{f}(t)\, \mathrm{d}t - \frac{1}{2s} \right]$$

$$\hat{f}(t) = \sum_{b=1}^{B} \tanh t L_b$$

$$\zeta_{\mathrm{P}}(s) = (2^{2s} - 1)\zeta_{\mathrm{R}}(2s) \sum_b \left(\frac{\pi}{L_b} \right)^{-2s}$$

$$\zeta'_{\mathrm{Im}}(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \log \left(\frac{\hat{f}(t)}{t} \right) \mathrm{d}t + \int_1^\infty \frac{\mathrm{d}}{\mathrm{d}t} \log \hat{f}(t)\, \mathrm{d}t$$

$$= -\log \mathcal{L} + \log B$$

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Spectral det of Neumann star

$$\det{}'(-\triangle) = \frac{2^B \mathcal{L}}{B}$$

Spectral det of star – mixed Dirichlet and Neumann nodes

$$\det{}'(-\triangle) = \frac{2^B}{B}\Big(\prod_d L_d\Big)\Big(\sum_d L_d^{-1}\Big)$$

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Theorem (Friedlander 06)

Spectral determinant for graph with Neumann matching.

$$\det{}'(-\triangle) = 2^B \frac{\mathcal{L}}{V} \frac{\prod_b L_b}{\prod_v d(v)} \det{}' R$$

$$R_{vw} = \begin{cases} -\sum_{b\in[v,w]} L_b^{-1} & v \neq w \\ \sum_{b\sim v} L_b^{-1} & v = w \end{cases}$$

Star graph:

$$\left(\prod_{b=1}^{B} L_b\right) \det{}' R = \det{}' \begin{pmatrix} \sum_{b=1}^{B} L_b^{-1} & -L_1^{-1} & -L_2^{-1} & \dots & -L_B^{-1} \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 1 \end{pmatrix} = 1$$

$$\det{}'(-\triangle) = 2^B \mathcal{L} / V B$$

Theorem (Spectral det of general star)

$$\det{}'(-\triangle) = \frac{2^B}{a_N} \det \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ I_B & \mathbf{L} \end{pmatrix}$$

Theorem (Spectral det of general graph)

$$\det{}'(-\triangle) = \frac{2^B}{a_N \prod_{b=1}^{B} L_b} \det \left(\mathbb{A} - \mathbb{B} \begin{pmatrix} \mathbf{L}^{-1} & -\mathbf{L}^{-1} \\ -\mathbf{L}^{-1} & \mathbf{L}^{-1} \end{pmatrix} \right)$$

e.g. star with Neumann center and Dirichlet nodes

$$\mathbb{A} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -1 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix} \qquad \mathbb{B} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathrm{I}_B & \mathbf{L} \end{pmatrix} = \det(\mathbb{A}\mathbf{L} - \mathbb{B}) = \det \begin{pmatrix} L_1 & -L_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & L_{B-1} & -L_B \\ -1 & \dots & -1 & -1 \end{pmatrix}$$

$$= L_1 \det \begin{pmatrix} L_2 & -L_3 & & 0 \\ & \ddots & \ddots & \\ 0 & & L_{B-1} & -L_B \\ -1 & \dots & -1 & -1 \end{pmatrix} - L_1^{-1} \prod_{b=1}^{B} L_b$$

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Iterating

$$\det \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathrm{I}_B & \mathbf{L} \end{pmatrix} = -\left(\prod_{b=1}^{B} L_b \right) \left(\sum_{b=1}^{B} L_b^{-1} \right) \tag{11}$$

To obtain t to infinity behavior of $\hat{f}$

$$\hat{f}(t) = \det \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathrm{I}_B & \frac{1}{t} \tanh(t\mathbf{L}) \end{pmatrix} = \frac{1}{t^{B-1}} \det \left(\mathbb{A} \tanh(t\mathbf{L}) - \mathbb{B} \right)$$

$$a_{B-1} = \det(\mathbb{A} - \mathbb{B}) = B \tag{12}$$

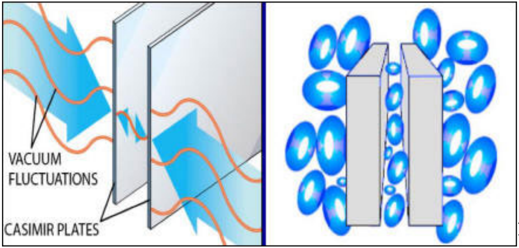
Spectral det of star with Dirichlet nodes

$$\det{}'(-\triangle) = \frac{2^B}{B} \left(\prod_{b=1}^{B} L_b \right) \left(\sum_{b=1}^{B} L_b^{-1} \right)$$

Casimir effect

(1948) Casimir predicts quantum mechanical attraction between uncharged plates.

(1997) Accurate observations of Casimir effect by Lamoreaux at Los Alamos and Mohideen and Roy at UC Riverside.



[1]

[1] Pictures from Wikipedia

Vacuum energy

- Casimir effect due to change in vacuum energy E_c.
- Formally vacuum energy $E = \frac{1}{2}\sum_{j=1}^{\infty} k_j$.
- Only changes in vacuum energy are observable.

Zeta fn regularization of vacuum energy (*Ray, Singer 71*)

$$E_c(t) = \frac{1}{2}\zeta(-1/2)$$

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Neumann star with equal bond lengths $L_b = L$

Zeta fn

$$\zeta(s) = \left(\frac{\pi}{L}\right)^{-2s} \left((B-1)2^{2s} - B + 2\right) \zeta_R(2s)$$

$$\begin{aligned} E_c &= \frac{1}{2}\zeta(-1/2) \\ &= \frac{\pi}{2L}\left(\frac{3}{2} - \frac{B}{2}\right)\zeta_{\mathrm{R}}(-1) \\ &= \frac{\pi}{48L}(B-3) \end{aligned} \tag{13}$$

(Agrees with Fulling, Kaplan, Wilson)

Incommensurate bond lengths

Vacuum energy

$$E_c = \frac{\pi}{48} \sum_b L_b^{-1} - \frac{1}{2\pi} \int_0^\infty t \frac{\hat{f}'(t)}{\hat{f}(t)} \, \mathrm{d}t$$

$$\hat{f}(t) = \sum_b \tanh tL_b$$

Setting $L_b = L$

$$E_c = \frac{\pi B}{48L} - \frac{L}{2\pi} \int_0^\infty t \frac{B \operatorname{sech}^2(tL)}{B \tanh(tL)} \, \mathrm{d}t \ .$$

Integrating again $E_c = \pi(B-3)/48L$.

Example II – vacuum energy for general star

Zeta fn of general star

$$\zeta(s) = \frac{\sin \pi s}{\pi} \left[\int_0^1 t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log \hat{f}(t)\,\mathrm{d}t + \int_1^\infty t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log(t^N \hat{f}(t))\,\mathrm{d}t \right]$$

$$- \frac{N \sin \pi s}{2\pi} + (2^{2s} - 1)\zeta_{\mathrm{R}}(2s) \sum_{b=1}^{B} \left(\frac{\pi}{L_b} \right)^{-2s} \qquad -\frac{1}{2} < s < 1$$

$$\hat{f}(t) = \det \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathrm{I}_B & \frac{1}{t} \tanh(t\mathbf{L}) \end{pmatrix}$$

- Continue $\zeta(s)$ to $s < -1/2$.
- $\hat{f}(t) \sim \frac{a_N}{t^N} + \frac{a_{N+j}}{t^{N+j}} + \cdots = \frac{a_N}{t^N} \left(1 + \frac{a_{N+j}}{a_N t^j} \right) + O\left(t^{-(N+j+1)} \right)$.
- $\log \hat{f}(t) \sim \log a_N - \log t^N + \frac{a_{N+j}}{a_N t^j} + O\left(t^{-(j+1)} \right)$.

Subtract leading and subleading order behavior.

Zeta fn of general star

$$\zeta(s) = \frac{\sin \pi s}{\pi}\left[\int_0^1 t^{-2s}\frac{\mathrm{d}}{\mathrm{d}t}\log \hat{f}(t)\,\mathrm{d}t - \frac{N}{2} + \frac{j\,a_{N+j}}{a_N\,(2s+j)} + \int_1^{\infty} t^{-2s}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\log(t^N\hat{f}(t)) - \frac{a_{N+j}}{a_N t^j}\right\}\mathrm{d}t\right] + (2^{2s}-1)\zeta_{\mathrm{R}}(2s)\sum_{b=1}^{B}\left(\frac{\pi}{L_b}\right)^{-2s} \qquad -\frac{j+1}{2} < s < 1$$

- If $j=1$ ζ divergent at $s=-1/2$.
- General graph vacuum energy also divergent.

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Regular part of vacuum energy of a general star

$$E_c = \frac{\pi}{48}\sum_{b=1}^{B} L_b^{-1} + \frac{N}{4\pi} - \frac{1}{2\pi}\left[\int_0^1 t\frac{\mathrm{d}}{\mathrm{d}t}\log \hat{f}(t)\,\mathrm{d}t \right.$$

$$\left. + \int_1^\infty t\frac{\mathrm{d}}{\mathrm{d}t}\left\{\log(t^N \hat{f}(t)) - \frac{a_{N+1}}{a_N t}\right\}\mathrm{d}t\right]$$

$$\hat{f}(t) = \det\begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathrm{I}_B & \frac{1}{t}\tanh(t\mathbf{L}) \end{pmatrix}$$

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Conclusions

- Obtained integral formulas for zeta fns of quantum graphs.
- Explicit formulas for Neumann star.
- New general formulation of spectral determinant.
- Integral formulation of vacuum energy.
- Heat kernel coefficients.