The Casimir effect for electromagnetic and semitransparent wedges: Breaking cylindrical symmetry

K. A. Milton

Collaboration: I. Brevik, S. Å. Ellingsen, K. Kirsten, and J. Wagner

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Wedge I

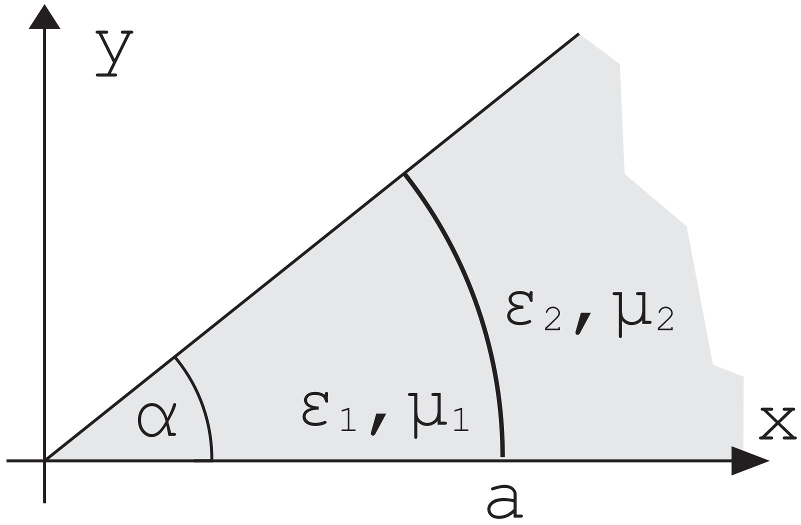


Figure 1: Perfectly conducting wedge, with a cylindrical perfectly conducting shell at radius a. Mostly, we consider that the indices of refraction are equal, $n^2 = \epsilon_1\mu_1 = \epsilon_2\mu_2$.

TE and TM polarizations

There are two polarizations:

1. TM polarization, which corresponds to

$$J_{mp}(\lambda_1 a) = 0,$$

   and

2. TE polarization, which corresponds to

$$J'_{mp}(\lambda_1 a) = 0$$

The electromagnetic field modes are proportional to

$$\cos mp\theta \quad \text{or} \quad \sin mp\theta, \quad p = \frac{\pi}{\alpha}.$$

Interior Casimir energy

The interior zero-point energy per unit length is

$$\mathcal{E}^{\rm int}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\sum_{s=1}^{\infty}\left[\omega_{0sk}^{\rm TE}+\sum_{m=1}^{\infty}(\omega_{msk}^{\rm TM}+\omega_{msk}^{\rm TE})\right].$$

TE zero mode, but no TM zero mode.

With the TE $m=0$ mode explicit,

$$\mathcal{E}^{\rm int}=-\frac{1}{8\pi n_1 a^2}\left\{\sum_{m=1}^{\infty}\int_0^\infty x^2 dx\left[\frac{I'_{mp}(x)}{I_{mp}(x)}+\frac{I''_{mp}(x)}{I'_{mp}(x)}\right]+\int_0^\infty dx\, x^2\frac{I''_0(x)}{I'_0(x)}\right\}.$$

Exterior and Interior Regions

The expression for the total energy is

$$\mathcal{E} = -\frac{1}{8\pi n a^2}\left\{\sum_{m=0}^{\infty}{}' \int_0^\infty x^2 dx \left[\frac{I'_{mp}(x)}{I_{mp}(x)} + \frac{I''_{mp}(x)}{I'_{mp}(x)} + \frac{K'_{mp}(x)}{K_{mp}(x)} + \frac{K''_{mp}(x)}{K'_{mp}(x)}\right] - \frac{1}{2}\int_0^\infty x^2 dx \frac{d}{dx}\ln\left(\frac{I_0(x)}{I'_0(x)}\frac{K_0(x)}{K'_0(x)}\right)\right\}$$

$$= \tilde{\mathcal{E}} + \hat{\mathcal{E}},$$

where $\tilde{\mathcal{E}}$ is finite but $\hat{\mathcal{E}}$ is divergent.

Elimination of Divergence

This zero-mode divergence is due to the sharp corners where the arc meets the wedge. We will proceed by setting this term aside, and computing the balance of the Casimir free energy. We note there is a closely related problem which Nesterenko et al. dubbed a cone. That is, we identify the two wedge boundaries at $\theta = 0$ and α, and impose periodic boundary conditions there. Thus we get precisely $2\tilde{\mathcal{E}}$ without the residual zero mode term $\hat{\mathcal{E}}$.

Finite part of free energy

After some manipulation, we obtain the following convenient form for the Casimir energy:

$$\tilde{\mathcal{E}} = \frac{1}{4\pi n a^2} \sum_{m=0}^{\infty}{}' \int_0^\infty x\,dx \ln\left[1 - x^2 \lambda_{mp}^2(x)\right].$$

where

$$\lambda_\nu(x) = (I_\nu(x) K_\nu(x))'.$$

Dielectric boundary at $r = a$

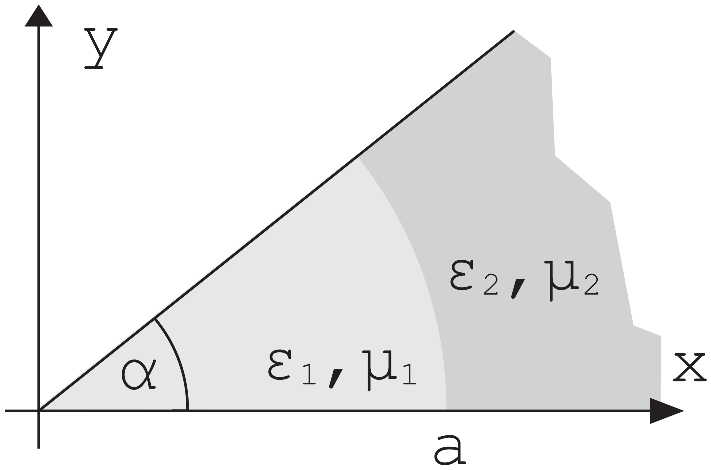


Figure 2: The wedge with a dielectric/diamagnetic boundary at $r = a$. The wedge boundaries are still perfectly conducting. Subsequently, we will allow $n_1 \neq n_2$.

Dispersion relation

$$\left[\frac{\mu_1}{u}\frac{J'_{mp}(u)}{J_{mp}(u)} - \frac{\mu_2}{v}\frac{H^{(1)'}_{mp}(v)}{H^{(1)}_{mp}(v)}\right]\left[\frac{\epsilon_1\omega^2}{u}\frac{J'_{mp}(u)}{J_{mp}(u)} - \frac{\epsilon_2\omega^2}{v}\frac{H^{(1)'}_{mp}(v)}{H^{(1)}_{mp}(v)}\right]$$

$$= m^2p^2k^2\left(\frac{1}{v^2} - \frac{1}{u^2}\right)^2,$$

where

$$u = \lambda_1 a, \quad v = \lambda_2 a, \quad \lambda_i^2 = n_i^2\omega^2 - k^2.$$

$n_1 = n_2$

The TE and TM modes do not decouple unless this condition is satisfied. The Casimir energy is then a generalization of the perfectly-conducting arc result:

$$\tilde{\mathcal{E}} = \frac{1}{4\pi n a^2} \sum_{m=0}^{\infty}{}' \int_0^\infty dx\, x \ln[1 - \xi^2 x^2 \lambda_{mp}^2],$$

where

$$\xi = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}.$$

Weak coupling, $\xi^2 \ll 1$

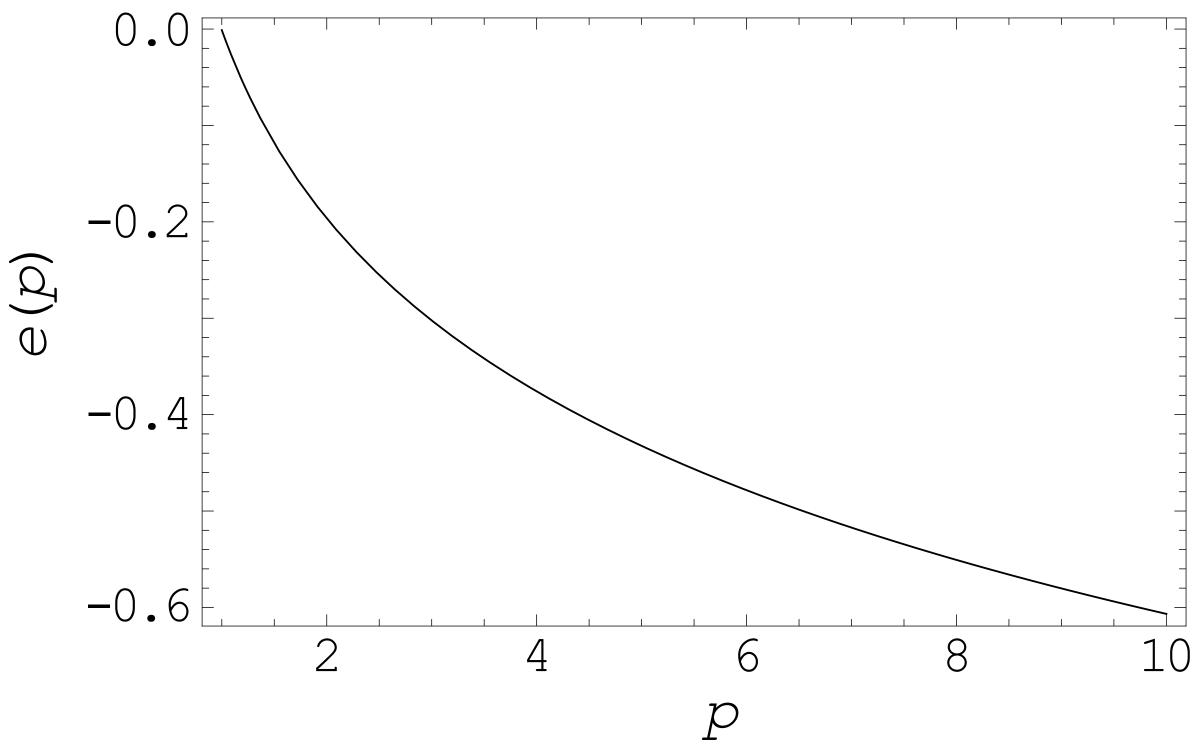


Figure 3: Casimir energy for weak coupling, $\xi^2 \ll 1$, as a function of $p = \pi/\alpha$.

Strong coupling, $\xi^2 = 1$

Perfectly conducting cylinder result reproduced for $\alpha = \pi$.

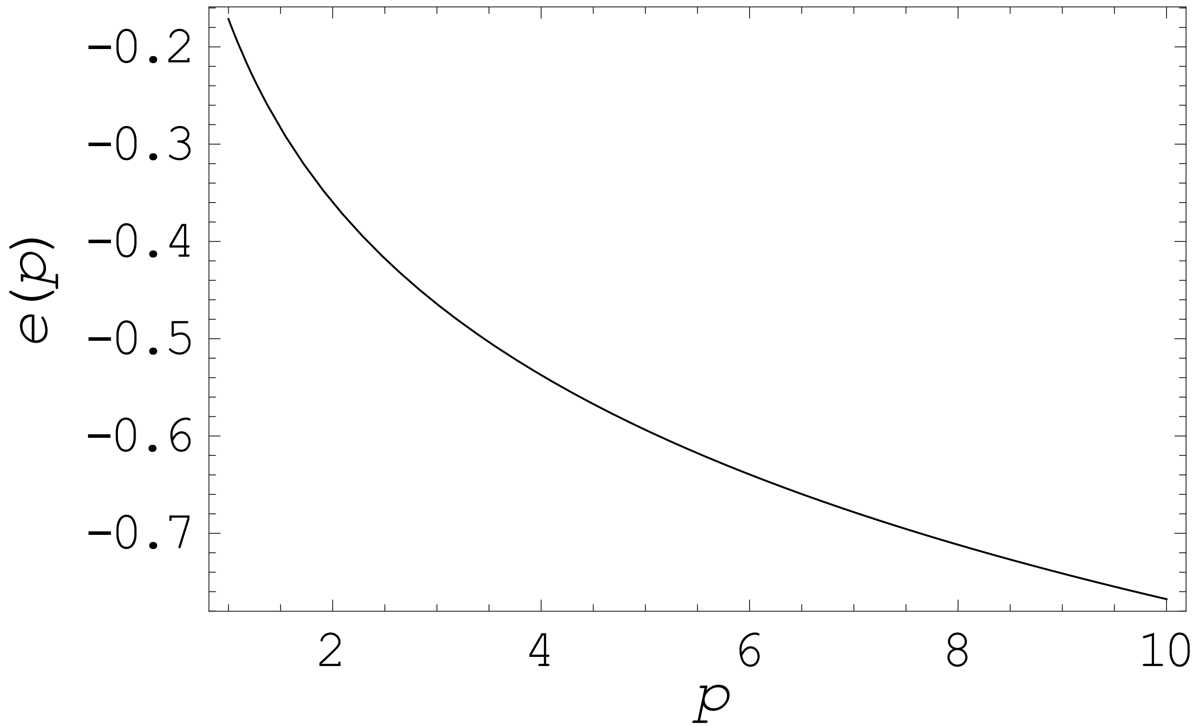


Figure 4: Casimir energy for $\xi^2 = 1$ vs. $p = \pi/\alpha$.

$n_1 \neq n_2, \mu_1 = \mu_2 = 1$

Only weak-coupling result is finite in this case.

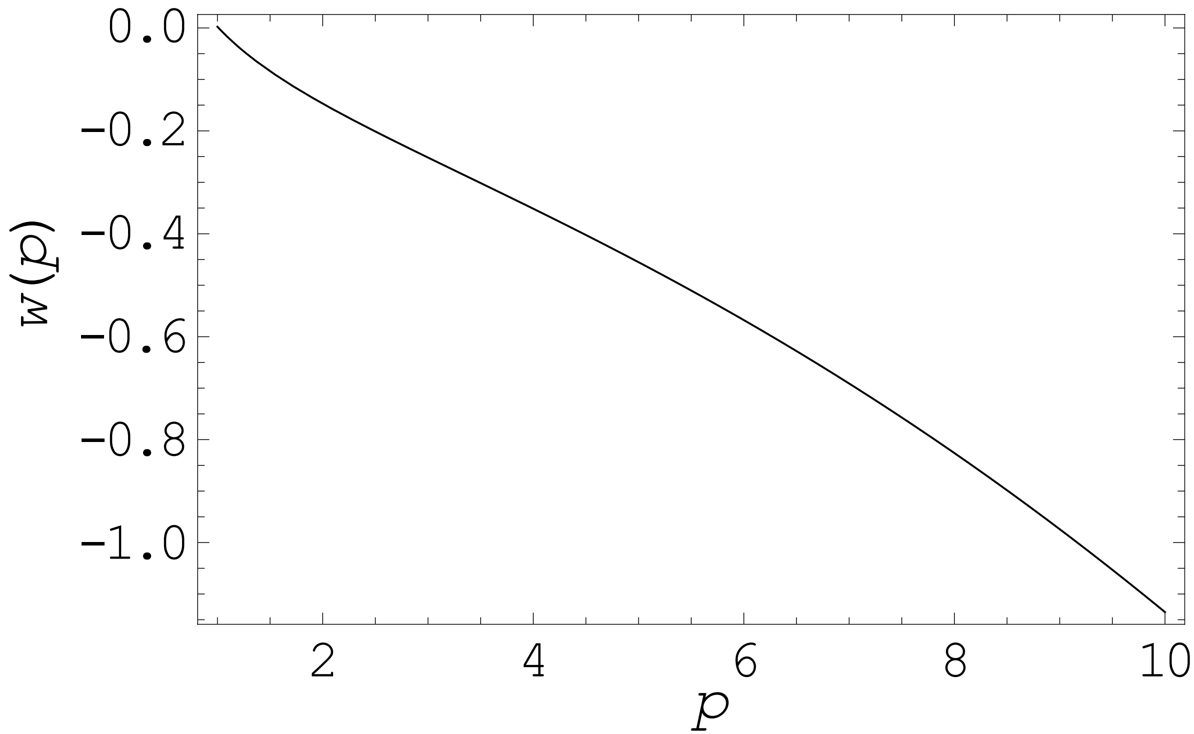


Figure 5: $\tilde{\mathcal{E}} = (\epsilon_1 - \epsilon_2)^2 w(p)/64\pi n a^2$, $|\epsilon_1 - \epsilon_2| \ll 1$, as a function of $p = \pi/\alpha$.

Wedge II

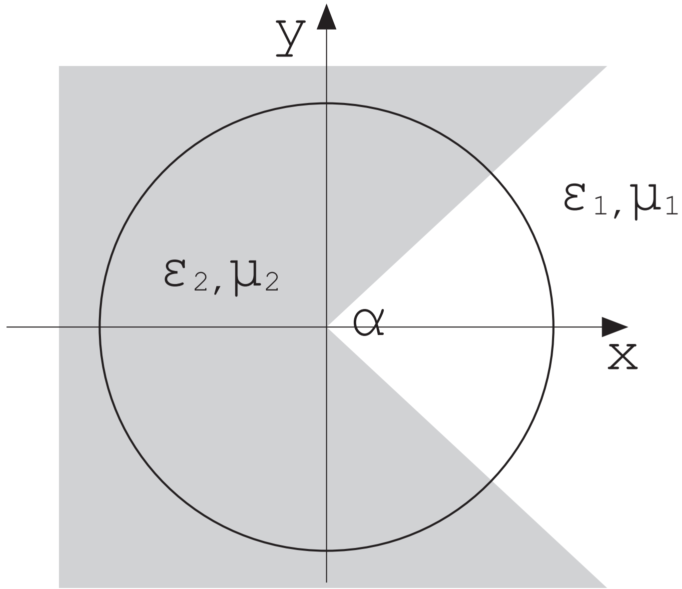


Figure 6: The wedge geometry considered.

Azimuthal Dispersion Relation

In this case there is a nontrivial dispersion relation for the azimuthal quantum number ν:

$$D(\nu,\omega) \equiv \sin^2(\nu\pi) - r^2 \sin^2(\nu(\pi-\alpha)) = 0.$$

The solutions are shown in the following figure:

ν as function of reflection coefficient

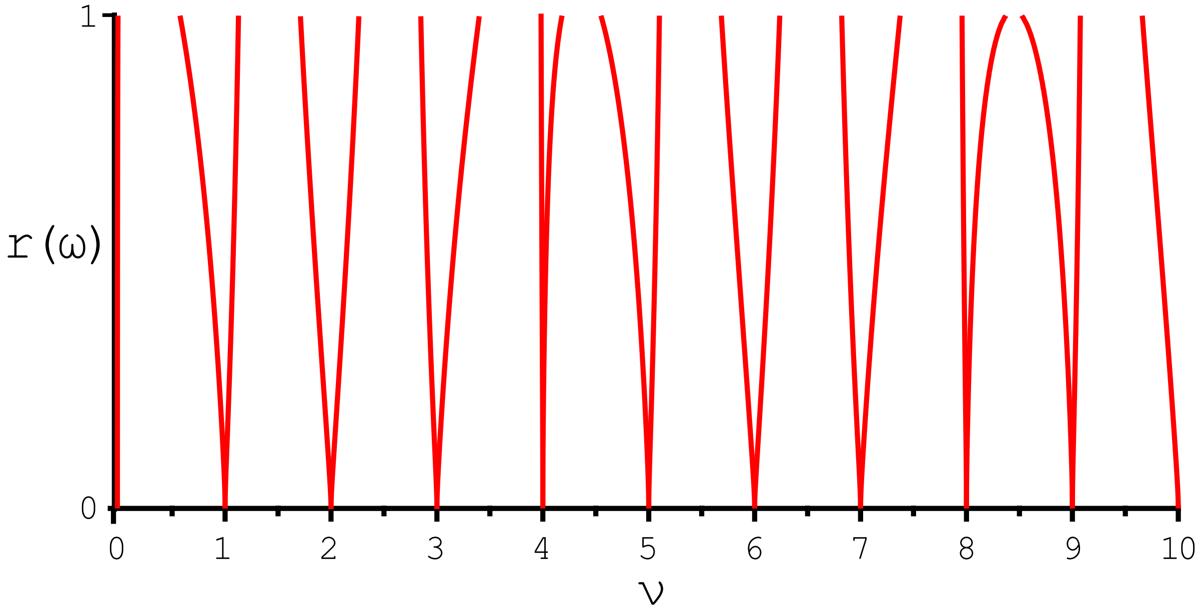


Figure 7: The solutions of the dispersion relation as a function of r and ν for $\alpha = 0.75$.

Argument principle gives CE

$$\tilde{\mathcal{E}} = \frac{1}{16\pi^3 i}\int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} d\zeta\zeta$$
$$\times \int_{-\infty}^{\infty} d\eta \left[\frac{d}{d\zeta}\ln g_{i\eta}(k_z, i\zeta)\right]\frac{d}{d\eta}\ln\frac{D(i\eta, i\zeta)}{D_0(i\eta)},$$

where

$$g_\nu(k_z,\omega) = 1 - x^2[(I_\nu(x)K_\nu(x))']^2,$$

and

$$D_0(\nu) = \sin^2\nu\pi.$$

Nondispersive approximation

$$\tilde{\mathcal{E}} = \frac{i}{8\pi^2 n a^2}\int_{-\infty}^{\infty} d\eta \frac{r^2 \sinh \eta(\pi-\alpha)}{\sinh \eta\pi[\sinh^2 \eta\pi - r^2 \sinh^2 \eta(\pi-\alpha)]}$$

$$\times[\alpha \sinh \eta(2\pi-\alpha) - (2\pi-\alpha)\sinh \eta\alpha]$$

$$\times \int_0^\infty dx x \ln[1 - x^2 \lambda_{i\eta}^2(x)].$$

Numerical Results

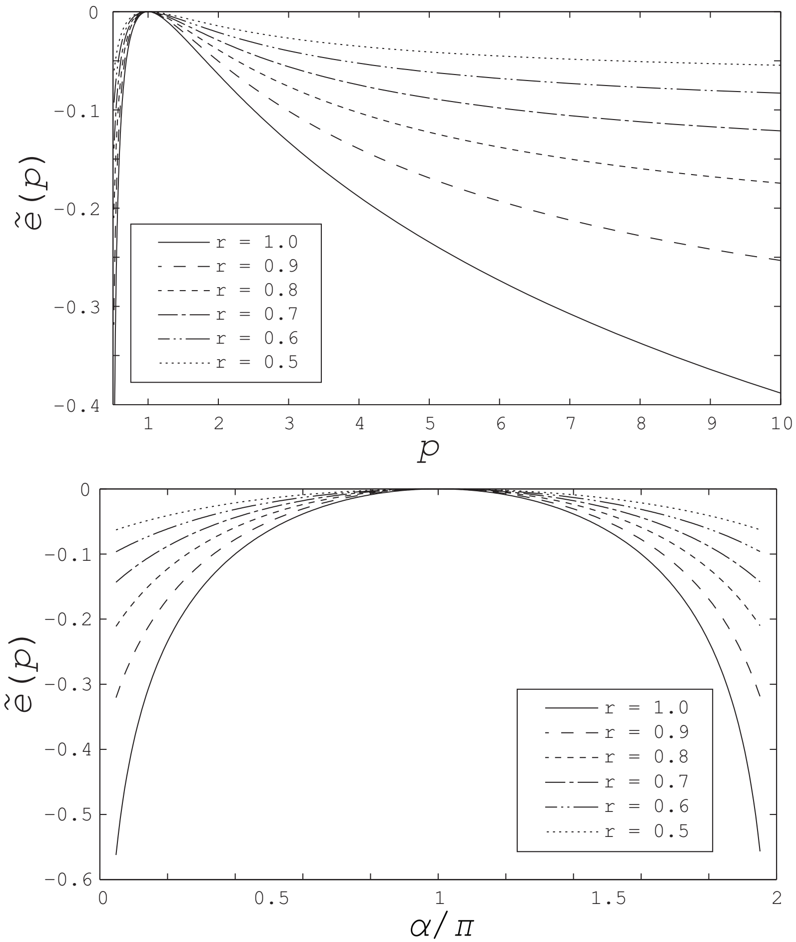
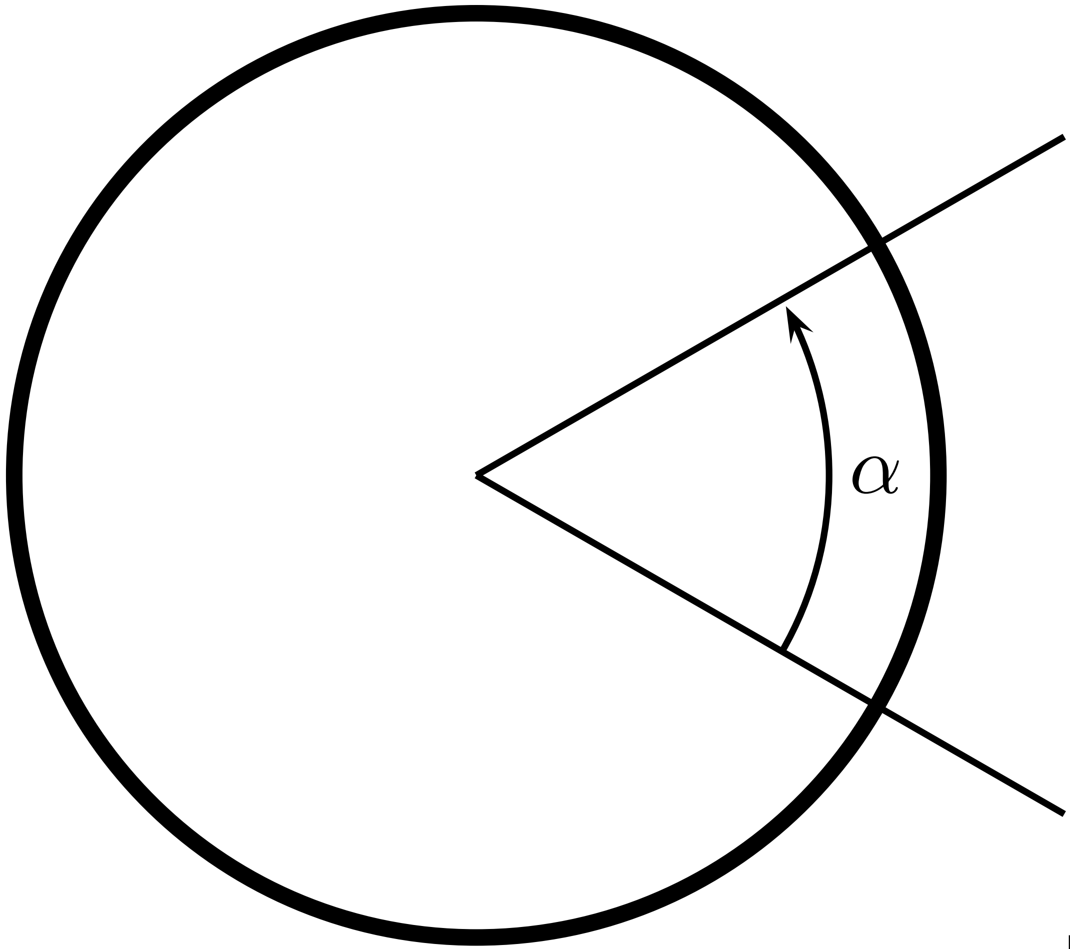


Figure 8: The function $\tilde{e}(p) = 8\pi n a^2 \tilde{\mathcal{E}}$

Wedge III



Semitransparent wedge

$$V(\rho,\theta) = V(\theta)/\rho^2, \quad V(\theta) = \lambda_1\delta(\theta-\alpha/2)+\lambda_2\delta(\theta+\alpha/2)$$

corresponds to the 2d Green's function satisfying

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \kappa^2 - \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} + \frac{V(\theta)}{\rho^2}\right] G(\rho,\theta;\rho',\theta')$$
$$= \frac{1}{\rho}\delta(\rho-\rho')\delta(\theta-\theta'),$$

where $\kappa^2 = k^2 - \omega^2$.

Conventional approach

$$G(\rho,\theta;\rho',\theta') = \sum_\nu \Theta_\nu(\theta)\Theta_\nu^*(\theta')g_\nu(\rho,\rho'),$$

where the reduced Green's function is

$$g_\nu(\rho,\rho') = I_\nu(\kappa\rho_<)K_\nu(\kappa\rho_>) - I_\nu(\kappa\rho)I_\nu(\kappa\rho')\frac{K_\nu(\kappa a)}{I_\nu(\kappa a)},$$

$$\rho,\rho' < a,$$

$$g_\nu(\rho,\rho') = I_\nu(\kappa\rho_<)K_\nu(\kappa\rho_>) - K_\nu(\kappa\rho)K_\nu(\kappa\rho')\frac{I_\nu(\kappa a)}{K_\nu(\kappa a)},$$

$$\rho,\rho' > a.$$

Dispersion relation

$$0 = D(\nu) = \sin^2 \nu(\alpha - \pi) - \left(1 - \frac{4\nu^2}{\lambda_1\lambda_2}\right) \sin^2 \pi\nu$$
$$- \left(\frac{\nu}{\lambda_1} + \frac{\nu}{\lambda_2}\right) \sin 2\pi\nu.$$

This agrees with the em wedge because here the reflection coefficient is $r_i = (1 + 2i\nu/\lambda_i)^{-1}$:

$$\Re\mathrm{e}\, r_1^{-1} r_2^{-1} = 1 - \frac{4\nu^2}{\lambda_1\lambda_2}, \quad \Im\mathrm{m}\, r_1^{-1} r_2^{-1} = \frac{2\nu}{\lambda_1} + \frac{2\nu}{\lambda_2}.$$

Here there is no $\nu = 0$ mode!

Casimir energy/length

$$\mathcal{E} = \frac{1}{2i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}2\omega^2\int_{-\infty}^{\infty}\frac{dk}{2\pi}\sum_\nu\int_0^\infty d\rho\,\rho\,g_\nu(\rho,\rho).$$

Using the argument principle, this becomes

$$\mathcal{E} = \frac{1}{8\pi^2 i}\int_0^\infty d\kappa\,\kappa^3\int_{-\infty}^{\infty}d\eta\left(\frac{d}{d\eta}\ln D(i\eta)\right)$$
$$\times\int_0^\infty d\rho\,\rho\,g_{i\eta}(\rho,\rho).$$

Removal of divergences

First subtract off the free radial Green's function without the circle at $\rho = a$,

$$\int_0^\infty d\rho\, \rho\, g_{i\eta}(\rho,\rho) \to \frac{a}{2\kappa}\frac{d}{d\kappa a}\ln[I_{i\eta}(\kappa a)K_{i\eta}(\kappa a)].$$

Remove the term present without the wedge potential:

$$D(\nu) \to \tilde{D}(\nu) = \frac{\lambda_1\lambda_2}{4\nu^2}\frac{D(\nu)}{\sin^2\pi\nu}.$$

Remove single plate energy

$$\tilde{D}(i\eta) \to \hat{D}(i\eta) = \frac{\tilde{D}(i\eta)}{\tilde{D}_1(i\eta)\tilde{D}_2(i\eta)}$$

$$= \frac{1}{(2\eta + \lambda_1 \coth \eta\pi)(2\eta + \lambda_2 \coth \eta\pi)}.$$

$$\times \Big[-\lambda_1\lambda_2 \sinh^2 \eta(\alpha - \pi)/\sinh^2 \eta\pi + 4\eta^2$$

$$+\lambda_1\lambda_2 + 2\eta(\lambda_1 + \lambda_2)\coth \eta\pi \Big]$$

Finite Casimir energy

This can be further simplified by noting that $\frac{d}{d\eta}\ln\hat{D}(i\eta)$ is odd, which then yields the expression

$$\mathcal{E} = -\frac{1}{4\pi^2 a^2}\int_0^\infty dx\, x\int_0^\infty d\eta\left(\frac{d}{d\eta}\ln\hat{D}(i\eta)\right)\arctan\frac{K_{i\eta}(x}{L_{i\eta}(x}$$

where

$$K_\mu(x) = \frac{\pi}{2\sin\pi\mu}\left[I_\mu(x) - I_{-\mu}(x)\right],$$

$$L_\mu(x) = \frac{i\pi}{2\sin\pi\mu}\left[I_\mu(x) + I_{-\mu}(x)\right],$$

Numerical evaluation

The difficulty numerically is that $K_{i\eta}(x)/L_{i\eta}(x)$ is an extremely oscillatory function of x for $x < \eta$, becoming infinitely oscillatory as $x \to 0$. For $x > \eta$, the ratio of modified Bessel functions of imaginary order monotonically and exponentially approaches zero. The function

$$h(\eta) = \int_0^\infty dx\, x^2 \frac{d}{dx} \arctan \frac{K_{i\eta}(x)}{L_{i\eta}(x)},$$

however, is very smooth.

Spline approximation

So to evaluate the double integral, we compute h at a finite number of discrete points, form a spline approximation which is indistinguishable from h, and then evaluate the function

$$e(\alpha) = \int_0^\infty d\eta\, h(\eta) \frac{d}{d\eta} \ln \hat{D}(i\eta),$$

numerically. The integrand here is quite strongly peaked in a neighborhood of the origin of size η. The Casimir energy is

$$\mathcal{E} = \frac{1}{8\pi^2 a^2} e(\alpha).$$

Equal couplings

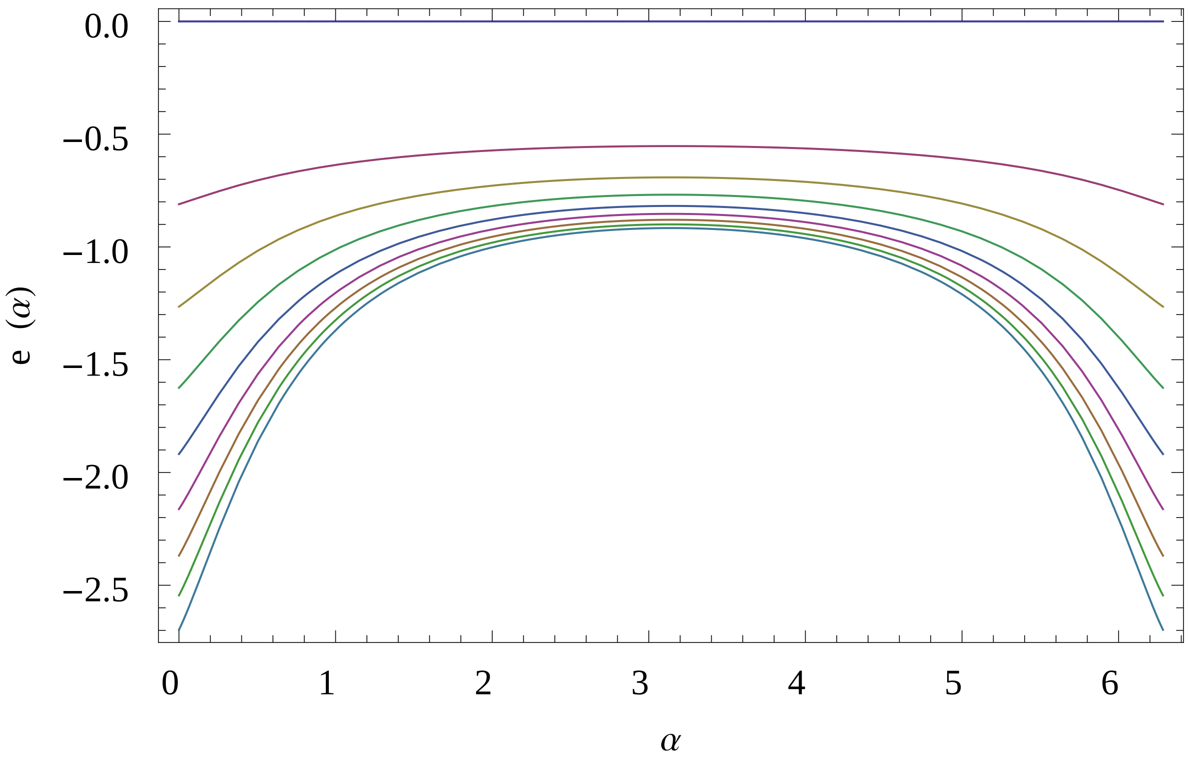


Figure 10: Casimir energies for $\lambda_1 = \lambda_2 = 0.5$ to 4.0, by steps of 0.5.

$\lambda_1 \neq \lambda_2$

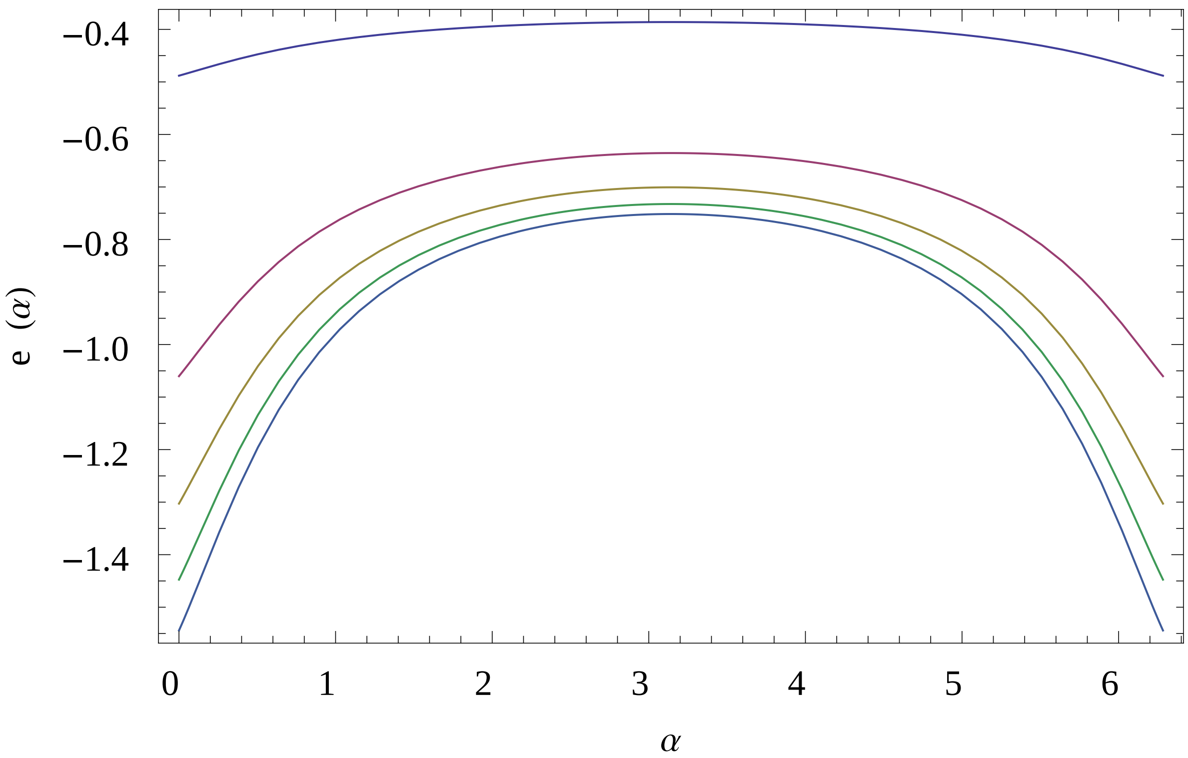


Figure 11: Casimir energies for $\lambda_1 = 1$ and $\lambda_2 = 0.1$ to 2.1, by steps of 0.5.