Semi Transparent Pistons

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Introduction

We begin considering the second order differential operator given by

$$L = -\frac{\partial^2}{\partial x^2} + \sigma\delta(x-a) \tag{1}$$

And the eigenvalue problem

$$L\mu_k = \lambda_k^2 \mu_k \tag{2}$$

where μ_k is continuous in $[0, L]$ and $\mu_k(0) = \mu_k(L) = 0$
Note that the eigenvalues are the λ_k^2 not the λ_k.

Solution to the Equation

Solving the differential equation for the region $0 < x < a$ gives

$$\mu_{1,k}(x) = A\sin(\lambda_k x) \tag{3}$$

and for the region $a < x < L$ we have

$$\mu_{2,k}(x) = B\sin(\lambda_k(L - x)) \tag{4}$$

We require the function μ_k to be continuous and to have a jump discontinuity at $x = a$ in the derivative due to the $\delta(x - a)$ term in the differential equation. We can restate this by

$$\begin{aligned} A\sin(\lambda_k a) &= B\sin(\lambda_k(L - a)) \\ -A\lambda_k\cos(\lambda_k a) - \lambda_k\cos(\lambda_k(L - a)) &= \sigma\mu(a) \end{aligned} \tag{5}$$

which, after normalizing to $\mu(a) = 1$ leds the eigenvalues to satisfy the equation

$$\sigma + \lambda\cot(\lambda a) + \lambda\cot(\lambda(L - a)) = 0 \tag{6}$$

Contour Integration

As usual, we define the associated Zeta function to this differential operator as

$$\zeta_L(s) = \sum_{k=1}^{\infty} \lambda_k^{-2s} \tag{7}$$

and our aim is to use the Cauchy's Residue Theorem (or the Argument Principle) to describe the Zeta function, so we look for a suitable function $g(\lambda)$ that has residue of 1 at every λ_k so that $\lambda^{-2s} g(\lambda)$ has residue and by Cauchy's Residue Theorem

$$\zeta_L(s) = \int_\gamma \lambda^{-2s} g(\lambda) d\lambda \tag{8}$$

for a suitable path γ.

Consider

$$F(\lambda) = \sigma + \lambda \cot(\lambda a) + \lambda \cot(\lambda(L - a)) \tag{9}$$

then, we have that

$$\frac{F'(\lambda)}{F(\lambda)} = \frac{d}{d\lambda} \ln(F(\lambda)) \tag{10}$$

has residue 1 at every λ_k, so we can use it for our goal for a γ enclosing the λ_k's. We can take γ to enclose the positive reals bigger or equal than the λ_k's, but this will count extra terms not originally wanted, i.e when

$$\lambda = \frac{n\pi}{a} \qquad \lambda = \frac{n\pi}{L - a} \tag{11}$$

so then, we have to substract back these extra contributions.

Convergence Region

For analyzing the region where the integral representation of the Zeta function is convergent, we discuss the behavior of F when $|\lambda| \rightarrow 0$ and $|\lambda| \rightarrow \infty$ for $\lambda \in \gamma$. Our ultimate goal is to deform the path into the imaginary axis, so we analyse $\lambda = ix$, for $x \in \mathbb{R}$. Thus F has the form

$$F(ix) = \sigma + x\coth(ax) + x\coth((L-a)x) \tag{12}$$

and hence, we have that for $x \rightarrow 0$

$$F(ix) \sim \sigma + \frac{1}{a} + \frac{1}{L-a} + \frac{L}{3}x^2 \tag{13}$$

and for $x \rightarrow \infty$

$$F(ix) \sim 2x + \sigma \tag{14}$$

Therefore for small x,the integral is well defined for

$$\mathrm{Re}(s) < 1 \tag{15}$$

while for large values of x, the integral converges for

$$\mathrm{Re}(s) > 0 \tag{16}$$

so the integral expression will converge for $0 < \mathrm{Re}(s) < 1$.

Contour Deformation

Now, we can deform the path γ to be the imaginary axis, since there are no other poles in the $\mathrm{Re}(s) \geq 0$ but since there is a pole at $\lambda = 0$, we have to analyze the behavior of the integral near zero, and for doing this, consider

$$\int_{C_\epsilon} d\lambda \lambda^{-2s} \frac{d}{d\lambda} \ln F(\lambda) \tag{17}$$

where C_ϵ is the circle given by $\lambda = \epsilon e^{i\theta}$, where $\pi/2 \leq \theta \leq \pi/2$.

The power series expansion for $F(\lambda)$ near zero is given by

$$F(\lambda) = \left(\sigma + \frac{1}{a} + \frac{1}{L-a}\right) - \frac{L}{3}\lambda^2 + O(\lambda^4) \tag{18}$$

and hence, the integral over C_ϵ is

$$\begin{aligned} &\int_{-\pi/2}^{\pi/2} d\theta \epsilon i e^{i\theta} \epsilon^{-2s} e^{-2si\theta} \frac{d}{d\epsilon e^{i\theta}} \ln F(\epsilon e^{i\theta}) \\ =& -ic\epsilon^{-2s+2} \frac{\sin((1-s)\pi)}{(1-s)} + O(\epsilon^2) \end{aligned} \tag{19}$$

where $c = \dfrac{2aL(L-a)}{3(L-a^2\sigma+aL\sigma)}$ and hence in $0 < \text{Re}(s) < 1$

$$\int_{C_\epsilon} d\lambda \lambda^{-2s} \frac{d}{d\lambda} \ln F(\lambda) \to 0 \tag{20}$$

as $\epsilon \longrightarrow 0$.

Phase conditions

Thus, we can deform γ to be the imaginary axis passing through zero. As λ approaches the positive imaginary axis, it has a phase of $e^{i\pi/2}$, thus, $\lambda^{-2s} = \left(e^{i\pi/2}x\right)^{-2s} = e^{-i\pi s}x^{-2s}$, where $x \in \mathbb{R}^+$. Likewise, for λ approaching the negative imaginary axis, the phase is $e^{-i\pi/2}$ and $\lambda^{-2s} = \left(e^{-i\pi/2}x\right)^{-2s} = e^{i\pi s}x^{-2s}$ for x a positive real.

Thus,

$$\int_\gamma d\lambda \lambda^{-2s} \frac{d}{d\lambda} \ln F(\lambda) = \int_\infty^0 dx e^{-i\pi s} x^{-2s} \frac{d}{dx} \ln F(ix) + \int_0^\infty dx e^{i\pi s} x^{-2s} \frac{d}{dx} \ln F(ix) = 2i \sin(\pi s) \int_0^\infty dx x^{-2s} \frac{d}{dx} \ln F(ix) \quad (21)$$

and the Zeta function is given by

$$\zeta_L(s) = \frac{\sin(\pi s)}{\pi} \int_0^\infty x^{-2s} dx \frac{d}{dx} \ln F(ix) - \text{extra contributions} \quad (22)$$

Zeta Function

The extra contributions are given by

$$\sum_{n=1}^{\infty}\left(\frac{\pi n}{a}\right)^{-2s}+\sum_{n=1}^{\infty}\left(\frac{\pi n}{L-a}\right)^{-2s}=\left(\left(\frac{\pi}{a}\right)^{-2s}+\left(\frac{\pi}{L-a}\right)^{-2s}\right)\zeta(2s) \tag{23}$$

and since $\ln(F(\lambda))$ has residue -1 at this values, we have that the Zeta function in the region $0 < \mathrm{Re}(s) < 1$ takes the form

$$\zeta_L(s)=\frac{\sin(\pi s)}{\pi}\int_0^{\infty}dx x^{-2s}\frac{d}{dx}\ln(F(ix)) + \left(\left(\frac{\pi}{a}\right)^{-2s}+\left(\frac{\pi}{L-a}\right)^{-2s}\right)\zeta(2s) \tag{24}$$

Analitic Continuation

We have to improve the behavior at infinity of the integrand, for this we use the asymptotic behavior at infinity

$$\begin{aligned}\int_0^\infty dx x^{-2s}\frac{d}{dx}\ln(F(ix)) &= \int_0^\infty dx x^{-2s}\frac{d}{dx}\ln\left(\frac{F(ix)}{2x+\sigma}\right)\\ &+\int_0^\infty dx x^{-2s}\frac{d}{dx}\ln(2x+\sigma)\\ &=\int_0^\infty dx x^{-2s}\frac{d}{dx}\ln\left(\frac{F(ix)}{2x+\sigma}\right)+\frac{2^{2s-1}\sigma^{-2s}\pi}{\sin(2\pi s)}\end{aligned} \quad (25)$$

for $0 < \mathrm{Re}(s) < 1/2$ and hence, doing the analytic continuation, the Zeta function can be written as

$$\zeta_L(s) = \frac{\sin(\pi s)}{\pi}\int_0^\infty dx x^{-2s}\frac{d}{dx}\ln\left(\frac{F(ix)}{2x+\sigma}\right)+\frac{2^{2s-1}\sigma^{-2s}}{\cos(\pi s)}$$

Operator Determinant

Evaluating the operator determinant taking the derivative with respect to s and the limit as $s \longrightarrow 0$,we have that

$$\zeta_L'(0) = -\ln\left(L + a(L-a)\sigma\right) - \ln 2 \tag{27}$$

Associated Force

We have that the associated force of the system is given by

$$F = -\frac{1}{2}\frac{\partial}{\partial a}\zeta_L(-1/2) \tag{28}$$

which gives

$$F = \frac{1}{2\pi}\frac{\partial}{\partial a}\int_0^\infty dxx\frac{d}{dx}\ln\left(\frac{F(ix)}{2x+\sigma}\right) - \frac{L(L-2a)\pi}{24a^2(L-a)^2} \tag{29}$$

which after applying integration by parts becomes

$$
\begin{aligned}
F =& \frac{1}{2\pi}\frac{\partial}{\partial a}\left(x\ln\left(\frac{F(ix)}{2x+\sigma}\right)\Bigg|_0^\infty - \int_0^\infty dx\ln\left(\frac{F(ix)}{2x+\sigma}\right)\right) \\
& - \frac{L(L-2a)\pi}{24a^2(L-a)^2} \\
=& -\frac{1}{2\pi}\frac{\partial}{\partial a}\left(\int_0^\infty dx\ln\left(\frac{F(ix)}{2x+\sigma}\right)\right) + \frac{1}{2\pi}\frac{\partial}{\partial a}\left(\frac{\pi^2 L}{12a(L-a)}\right) \\
=& -\frac{1}{2\pi}\frac{\partial}{\partial a}\left(\int_0^\infty dx\ln\left(\frac{F(ix)}{2x+\sigma}\right)\right) \\
& + \frac{1}{2\pi}\frac{\partial}{\partial a}\int_0^\infty \ln\left(\coth\left(\frac{3a(L-a)}{2L}x\right)\right)dx
\end{aligned}
\tag{30}
$$

So the force is given by

$$F = -\frac{1}{2\pi}\frac{\partial}{\partial a}\int_0^\infty \ln\left(\frac{\sigma + x\coth(ax) + x\coth((L-a)x)}{(2x+\sigma)\coth\left(\frac{3a(L-a)}{2L}x\right)}\right) dx \tag{31}$$

and when doing the numerical approach we have the following graph

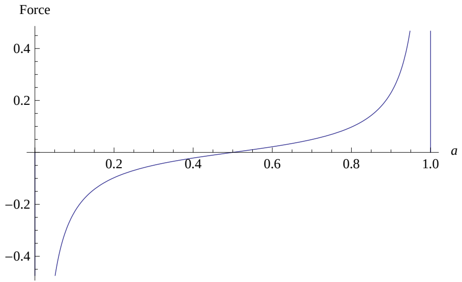

Figure: Casimir Force

Trying to determine the sign of the force analitically, since the are two oposite terms , one due to the contour integration, and the other one because of the extra contributions which behave like

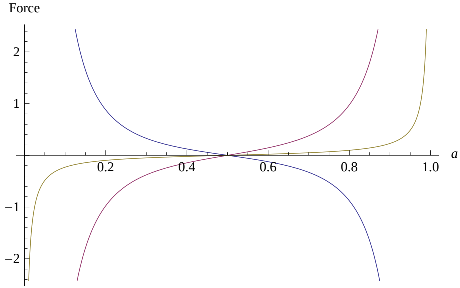

Figure: Individual Terms

And the integrand behavior is also oscillatory, it does not have a the same sign in each half of the interval

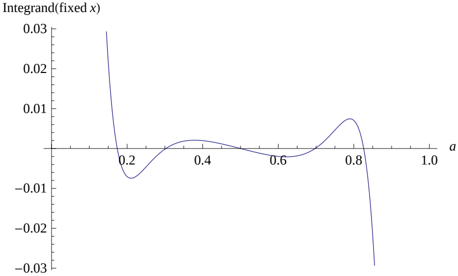

Figure: Integrand Behavior

And we have that the energy will have this shape

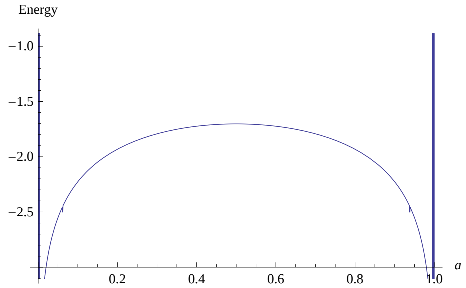

Figure: Casimir Energy

Two Dimensional Case

Like in the previous consideration, we start analyzing the second order differential operator

$$L = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \sigma\delta(x - a) \tag{32}$$

which has eigenvalues λ_k^2 and eigenfunctions μ_{λ_k}

$$L\mu_{\lambda_k} = \lambda_k^2 \mu_{\lambda_k} \tag{33}$$

Requiring also the continuity and the jump of the derivative at $x = a$ leads to a solution

$$\mu_{1,\lambda_k} = A\sin(\sqrt{\lambda_k^2 - C^2}x)\sin(Cy)$$
$$\mu_{2,\lambda_k} = B\sin(\sqrt{\lambda_k^2 - C^2}(L-x))\sin(Cy) \tag{34}$$

where

$$C = \frac{\pi n}{M} \tag{35}$$

where $n \in \mathbb{N}$ and each n defines a mode in the solution

Associated Zeta Function

As before, after normalizing the constants, we have that the λ_k's satisfy the equation

$$F(\nu) = \sigma + \nu \cot(\nu a) + \nu \cot(\nu(L - a)) = 0 \qquad (36)$$

where

$$\lambda^2 = \nu^2 + \frac{\pi}{M} n^2 \qquad (37)$$

hence, the associated zeta function can be written as

$$\zeta_L(s) = \sum_{k=1}^{\infty} \lambda_k^{-2s} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\nu_m^2 + \left(\frac{\pi}{M} n \right)^2 \right)^{-s} \qquad (38)$$

Contour Integration

Therefore, by the discussion in the previous section, we have that

$$\zeta_L(s) = \frac{1}{2\pi i} \sum_{m=1}^{\infty} \int_\gamma d\nu \left(\nu^2 + \left(\frac{\pi}{M} n \right)^2 \right) \frac{d}{d\nu} \ln \left(F(\nu) \right) \qquad (39)$$

where γ is a path that encloses the values of ν_m but misses where F is not defined.

For a fixed n, the expression

$$\int_\gamma d\nu \left(\nu^2 + \left(\frac{\pi}{M} n\right)^2\right) \frac{d}{d\nu} \ln\left(F(\nu)\right) \tag{40}$$

converges when $\mathrm{Re}(s) > 0$. Now the behavior near zero does not affect the convergence of (40), but the behavior near $i\pi n/M$ does. Near this, (40) will converge for $\mathrm{Re}(s) < 1/2$.

As before, for C_ϵ being the half circle $i\pi n/M + \epsilon e^{i\theta}$, for $-\pi/2 \leq \theta \leq \pi/2$, we have that

$$\int_{C_\epsilon} d\nu \left(\nu^2 + \left(\frac{\pi}{M} n\right)^2\right)^{-s} \frac{d}{d\nu} \ln\left(F(\nu)\right) \to 0 \tag{41}$$

as $\epsilon \to 0$

Phase Conditions

Thus, we can deform γ to be the imaginary axis passing thru $\pm i\pi n/M$. For $\nu = xe^{i\pi/2}$, where $x \in \mathbb{R}^+$, we have that

$$\left(\nu^2 + \left(\frac{\pi n}{M}\right)^2\right)^{-s} = \left(\left(\frac{\pi n}{M}\right)^2 - x^2\right)^{-s} \tag{42}$$

which is real for $0 < x < \pi n/M$ and has a phase of $\left(e^{i\pi}\right)^{-s} = e^{-i\pi s}$ for $x > \pi n/M$.
Similarly, for $\nu = xe^{-i\pi/2}$, we have that it real for $0 < x < \pi n/M$ and has a phase of $\left(e^{-i\pi}\right)^{-s} = e^{i\pi s}$ for $x > \pi n/M$.

Therefore, we have that

$$\begin{aligned}
&\int_\gamma d\nu \left(\nu^2 + \left(\frac{\pi n}{M}\right)^2\right) \frac{d}{d\nu} \ln\left(F(\nu)\right) \\
=&\int_\infty^{\pi n/M} dx e^{-i\pi s} \left(x^2 - \left(\frac{\pi n}{M}\right)^2\right)^{-s} \frac{d}{dx} \ln(F(ix)) \\
&+ \int_{\pi n/M}^{0} dx \left(\left(\frac{\pi n}{M}\right)^2 - x^2\right)^{-s} \frac{d}{dx} \ln(F(ix)) \\
&+ \int_0^{\pi n/M} dx \left(\left(\frac{\pi n}{M}\right)^2 - x^2\right)^{-s} \frac{d}{dx} \ln(F(ix)) \\
&+ \int_{\pi n/M}^{\infty} dx e^{i\pi s} \left(x^2 - \left(\frac{\pi n}{M}\right)^2\right)^{-s} \frac{d}{dx} \ln(F(ix)) \\
=&2i\sin(\pi s) \int_{\pi n/M}^{\infty} dx \left(x^2 - \left(\frac{\pi n}{M}\right)^2\right)^{-s} \frac{d}{dx} \ln(F(ix))
\end{aligned} \tag{43}$$

and therefore the Zeta function takes the form

$$\zeta_L(s) = \frac{\sin(\pi s)}{\pi} \sum_{n=1}^{\infty} \int_{\pi n/M}^{\infty} dx \left(x^2 - \left(\frac{\pi n}{M}\right)^2 \right)^{-s} \frac{d}{dx} \ln(F(ix)) - \text{extra contributions} \quad (44)$$

where the extra contributions are

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \left(\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{M}\right)^2 \right)^{-s} + \sum_{m=1}^{\infty} \left(\left(\frac{\pi m}{L-a}\right)^2 + \left(\frac{\pi n}{M}\right)^2 \right)^{-s} \right) = E\left(s, \frac{\pi^2}{a^2}, \frac{\pi^2}{M^2}\right) + E\left(s, \frac{\pi^2}{(L-a)^2}, \frac{\pi^2}{M^2}\right) \quad (45)$$

where E is the Epstein Zeta function.

Hence, the Zeta function takes the form

$$\zeta_L(s) = \frac{\sin(\pi s)}{\pi} \sum_{n=1}^{\infty} \int_{\pi n/M}^{\infty} dx \left(x^2 - \left(\frac{\pi n}{M} \right)^2 \right)^{-s} \frac{d}{dx} \ln(F(ix)) + E\left(s, \frac{\pi^2}{a^2}, \frac{\pi^2}{M^2} \right) + E\left(s, \frac{\pi^2}{(L-a)^2}, \frac{\pi^2}{M^2} \right) \quad (46)$$

Behavior Improvement

For improving the convergence of the integral, we can use the same trick as before, considering the asymptotic behavior of F

$$\int_{\pi n/M}^{\infty} dx \left(x^2 - \left(\frac{\pi n}{M}\right)^2\right)^{-s} \frac{d}{dx} \ln\left(\frac{F(ix)}{2x+\sigma}\right) + \int_{\pi n/M}^{\infty} dx \left(x^2 - \left(\frac{\pi n}{M}\right)^2\right)^{-s} \frac{d}{dx} \ln(2x+\sigma) \quad (47)$$

but the second integral is a little hard to handle, so instead, consider the power series expantion of the asymptotic behavior at infinity

$$\frac{1}{2x+\sigma} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sigma^{k-1}}{2^k} x^{-k} \quad (48)$$

Instead of using the $2x + \sigma$ asymptote, we start substracting a couple of terms of the series expantion, so we have that for $-1/2 < \mathrm{Re}(s)$

$$\zeta_L(s) = \frac{\sin(\pi s)}{\pi} \sum_{n=1}^{\infty} \int_{\pi n/M}^{\infty} dx \left(x^2 - \left(\frac{\pi n}{M} \right)^2 \right)^{-s} \times \left(\frac{d}{dx} \ln F(ix) - \left(x^{-1} - \frac{\sigma x^{-2}}{2} \right) \right)$$
$$+ \left(\frac{\pi}{M} \right)^{-2s} \zeta(2s) - \frac{\sigma \sin(\pi s)}{2\pi} \left(\frac{\pi}{M} \right)^{-2s-1} \zeta(2s+1) B(1-s, s+1/2)$$
$$+ E\left(s, \frac{\pi^2}{a^2}, \frac{\pi^2}{M^2} \right) + E\left(s, \frac{\pi^2}{(L-a)^2}, \frac{\pi^2}{M^2} \right) \quad (49)$$

Operator Determinant

Therefore, it is possible to evaluate the operator determinant that is

$$\zeta_L'(0) = \sum_{n=1}^{\infty}\left(\ln\left(F\left(i\frac{\pi n}{M}\right)\right) - \ln\left(\frac{\pi n}{M}\right) - \frac{\sigma}{2}\left(\frac{M}{\pi n}\right)\right) - \ln(2M)$$
$$- \frac{\sigma M}{2\pi}\left(3\gamma + 2\ln(M) - 2\ln(\pi) + \frac{1}{\sqrt{\pi}}\Gamma'(1/2)\right)$$
$$+ \sum_{n=1}^{\infty}\ln\left(1 - e^{-\frac{2Mn}{a}}\right) + \frac{M\pi}{12a} + \frac{a^2\ln(2\pi)}{2\pi^2}$$
$$+ \sum_{n=1}^{\infty}\ln\left(1 - e^{-\frac{2Mn}{L-a}}\right) + \frac{M\pi}{12(L-a)} + \frac{(L-a)^2\ln(2\pi)}{2\pi^2} \tag{50}$$

Casimir Force

For calculating the behavior at $s = -1/2$, we need to consider some extra terms so that the Zeta function converge at $s = -1/2$. Taking the asymptotic expantion of the infitity behavior of F up to 3 terms gives

$$\zeta_L(s) = \frac{\sin(\pi s)}{\pi} \sum_{n=1}^{\infty} \int_{\pi n/M}^{\infty} dx \left(x^2 - \left(\frac{\pi n}{M} \right)^2 \right)^{-s} \left(\frac{d}{dx} \ln F(ix) - \left(x^{-1} \right.\right.$$

$$+ \left(\frac{\pi}{M} \right)^{-2s} \zeta(2s) - \frac{\sigma \sin(\pi s)}{2\pi} \left(\frac{\pi}{M} \right)^{-2s-1} \zeta(2s+1) B(1-s, s+1/2)$$

$$+ \left(\frac{\pi}{M} \right)^{-2(s+1)} \zeta(2s+2) s + E\left(s, \frac{\pi^2}{a^2}, \frac{\pi^2}{M^2} \right) + E\left(s, \frac{\pi^2}{(L-a)^2}, \frac{\pi^2}{M^2} \right) \tag{51}$$

and hence the force is given by

$$
\begin{aligned}
F = -\frac{1}{2}\frac{\partial}{\partial a}\zeta_L(-1/2) = \frac{1}{2}\frac{\partial}{\partial a}\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{\pi n/M}^{\infty} dx \left(x^2 - \left(\frac{\pi n}{M}\right)^2\right)^{1/2} \\
\times \left(\frac{d}{dx}\ln F(ix) - \left(x^{-1} - \frac{\sigma x^{-2}}{2} + \frac{\sigma^2}{4}x^{-3}\right)\right) \\
- \frac{1}{2}\frac{\partial}{\partial a}E\left(-\frac{1}{2}, \frac{\pi^2}{a^2}, \frac{\pi^2}{M^2}\right) - \frac{1}{2}\frac{\partial}{\partial a}E\left(-\frac{1}{2}, \frac{\pi^2}{(L-a)^2}, \frac{\pi^2}{M^2}\right) \quad (52)
\end{aligned}
$$