Hertz potentials in curvilinear coordinates

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Quantum Vacuum Workshop

Purpose and Outline

Purpose

- To describe any arbitrary electromagnetic field in a bounded geometry in terms of two scalar fields, and
- To define these fields such that the boundary conditions consist of at most first-derivatives of the fields.

Outline

1. Review of Electromagnetism and Hertz Potentials in Vector Formalism
2. Overview of Differential Form Formalism
3. Formulation of Electromagnetism in Differential Form Formalism
4. “Scalar” Hertz Potential Examples

Maxwell's Equations

Vector Equations

$$\vec{\nabla} \cdot \vec{E} = \rho \quad (1)$$

$$\vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{\jmath} \quad (2)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (3)$$

$$\partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0 \quad (4)$$

Constants

For simplicity, take

$$\epsilon_0 = \mu_0 = c = 1.$$

Potentials

(3) and (4) imply

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad (5)$$

$$\vec{E} = -\vec{\nabla} V - \partial_t \vec{A} \quad (6)$$

Charge Conservation

Also note that (1) and (2) imply $\partial_t \rho + \vec{\nabla} \cdot \vec{\jmath} = 0$.

Hertz Potentials

Hertz Potentials

$$V = -\vec{\nabla} \cdot \vec{\Pi}_e \tag{7}$$

$$\vec{A} = \partial_t \vec{\Pi}_e + \vec{\nabla} \times \vec{\Pi}_m \tag{8}$$

Lorenz Condition

$$\partial_t V + \vec{\nabla} \cdot \vec{A} = 0$$

Inhomogeneous Maxwell Equations

$$\vec{\nabla} \cdot (\Box \vec{\Pi}_e) = \rho \tag{9}$$

$$\vec{\nabla} \times (\Box \vec{\Pi}_m) + \partial_t (\Box \vec{\Pi}_m) = \vec{\jmath} \tag{10}$$

Definition

$$\Box = \partial_t^2 - \nabla^2$$

Hertz Potentials

From here on, set $\rho = 0, \vec{\jmath} = \vec{0}$.

Equations of Motion

$$\Box \vec{\Pi}_e = \vec{\nabla} \times \vec{W} + \vec{\nabla} g + \partial_t \vec{G} \tag{11}$$

$$\Box \vec{\Pi}_m = -\partial_t \vec{W} - \vec{\nabla} w + \vec{\nabla} \times \vec{G} \tag{12}$$

The w and $\vec{W}$ terms come from (9) and (10) just as V and $\vec{A}$ came from (3) and (4). The g and $\vec{G}$ terms come from relaxing the Lorenz condition.

Differential Geometry

Let $(x^0, \ldots, x^{n-1})$ be the coordinate system on an n-dimensional manifold. Then we write vectors on that manifold as

$$\vec{v} = v^0 \partial_{x^0} + \cdots + v^{n-1} \partial_{x^{n-1}},$$

and 1-forms (or covectors) as

$$v = v_0 dx^0 + \cdots v_{n-1} dx^{n-1}.$$

Example

For Minwoski space, we can write the electromagnetic potential A^μ as the vector

$$\vec{A} = A^\mu \partial_{x^\mu} = V \partial_t + A^x \partial_x + A^y \partial_y + A^z \partial_z$$

(not to be confused with the 3-vector from before) or as the 1-form

$$A = -V dt + A_x dx + A_y dy + A_z dz.$$

Differential Geometry

Definition

The **wedge product** of two forms, written $f \wedge g$, is the antisymmetrized tensor product.

Example

$$dx \wedge dy = dx \otimes dy - dy \otimes dx$$

$$dx \wedge dy \wedge dz = dx \otimes dy \otimes dz - dx \otimes dz \otimes dy + dz \otimes dx \otimes dy + \ldots$$

Definition

For a k-form of the form $f = f_{\alpha_1 \cdots \alpha_k} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}$, define the **differential of** f as

$$df = \sum_{\mu=0}^{n-1} \frac{\partial f_{\alpha_1 \cdots \alpha_k}}{\partial x^i} dx^{\mu} \wedge dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}.$$

Differential Geometry

Definition

Let $\eta_{0\cdots n}$ be the volume form. For a k-form of the form $f = f_{\alpha_1\cdots\alpha_k} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}$, define the **Hodge dual of** f as

$$*f = f_{\alpha_1\cdots\alpha_k} \eta^{\alpha_1\cdots\alpha_k}{}_{\beta_1\cdots\beta_{n-k}} dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_{n-k}}.$$

Definition

$$\delta = *d*$$

Definition

$$\Box = d\delta + \delta d = d*d* + *d*d$$

Electromagnetism Revisited

Maxwell's Equations

Define

$$F = -E_i dt \wedge dx^i + B_i * (dt \wedge dx^i).$$

Then (1) – (4) become

$$\delta F = J \tag{13}$$

$$dF = 0 \tag{14}$$

Potentials

(14) implies

$$F = dA \tag{15}$$

Lorenz Condition

$$\delta A = 0$$

Relaxed Lorenz Condition

$$\delta(A + G) = 0$$

Hertz Potentials

The relaxed Lorenz condition implies

$$A = \delta \Pi - G \tag{16}$$

Electromagnetism Revisited

Since

$$0 = J = \delta F = \delta dA = \delta d(\delta \Pi - G) \tag{17}$$

$$= \delta(\Box \Pi - dG), \tag{18}$$

we can write

Equations of Motion

$$\Box \Pi = dG + \delta W. \tag{19}$$

Cartesian Coordinates

$$(x^0, x^1, x^2, x^3) = (t, x, y, z)$$

$$\begin{aligned}\Pi &= \phi dt \wedge dz + \psi * (dt \wedge dz) \\ &= \phi dt \wedge dz + \psi dx \wedge dy\end{aligned}$$

Equations of Motion

Given $\Box \Pi = 0$,

$$\begin{aligned}\Box \phi &= 0 \\ \Box \psi &= 0\end{aligned}$$

Axial Cylindrical Coordinates

$$(x^0, x^1, x^2, x^3) = (t, \rho, \varphi, z)$$

$$\begin{aligned} \Pi &= \phi dt \wedge dz + \psi * (dt \wedge dz) \\ &= \phi dt \wedge dz + \rho \psi d\rho \wedge d\varphi \end{aligned}$$

Equations of Motion

Given $\Box \Pi = 0$,

$$\Box \phi = 0$$
$$\Box \psi = 0$$

Spherical Coordinates

$$(x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$$

$$\begin{aligned}\Pi &= \phi dt \wedge dr + \psi * (dt \wedge dr) \\ &= \phi dt \wedge dr + \psi r^2 \sin\theta d\theta \wedge d\varphi\end{aligned}$$

Definition

$$\hat{\Box} = \Box + \tfrac{2}{r}\partial_r = \partial_t^2 - \partial_r^2 - \tfrac{1}{r^2 \sin\theta}\partial_\theta \sin\theta \partial_\theta - \tfrac{1}{r^2 \sin^2\theta}\partial_\varphi^2$$

Equations of Motion?

$$\begin{aligned}\Box\Pi = {} & (\hat{\Box}\phi - \partial_r \frac{2\phi}{r}) dt \wedge dr - \partial_\theta \frac{2\phi}{r} dt \wedge d\theta - \partial_\varphi \frac{2\phi}{r} dt \wedge d\varphi \\ & + (\hat{\Box}\psi - \partial_r \frac{2\psi}{r}) * (dt \wedge dr) - \partial_\theta \frac{2\psi}{r} * (dt \wedge d\theta) - \partial_\varphi \frac{2\psi}{r} * (dt \wedge d\varphi)\end{aligned}$$

Spherical Coordinates

$$G = \frac{2}{r}\phi, \quad dG = -\partial_r \frac{2\phi}{r} dt \wedge dr - \partial_\theta \frac{2\phi}{r} dt \wedge d\theta - \partial_\varphi \frac{2\phi}{r} dt \wedge d\varphi$$

$$*W = \frac{2}{r}\psi, \quad \delta W = -\partial_r \frac{2\psi}{r} *(dt \wedge dr) - \partial_\theta \frac{2\psi}{r} *(dt \wedge d\theta) - \partial_\varphi \frac{2\psi}{r} *(dt \wedge d\varphi)$$

Equations of Motion

Given $\Box \Pi = dG + \delta W$,

$$\hat{\Box}\phi = 0$$

$$\hat{\Box}\psi = 0$$

Schwarzchild Coordinates

$$(x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$$
$$ds^2 = (1 - \tfrac{r_s}{r})dt^2 + \tfrac{1}{(1-\frac{r_s}{r})}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

$$\Pi = \phi dt \wedge dr + \psi * (dt \wedge dr)$$
$$= \phi dt \wedge dr + \psi r^2 \sin\theta d\theta \wedge d\varphi$$

$$G = \tfrac{2\zeta}{r}\phi$$
$$*W = \tfrac{2\zeta}{r}\psi$$

Definition

$$\zeta = 1 - \tfrac{r_s}{r}$$

$$\hat{\Box} = \tfrac{1}{\zeta}\partial_t^2 - \partial_r \zeta \partial_r - \tfrac{1}{r^2 \sin\theta}\partial_\theta \sin\theta \partial_\theta - \tfrac{1}{r^2 \sin^2\theta}\partial_\varphi^2$$

Equations of Motion

Given $\Box\Pi = dG + \delta W$,

$$\hat{\Box}\phi = 0$$
$$\hat{\Box}\psi = 0$$

Radial Cylindrical Coordinates

$$(x^0, x^1, x^2, x^3) = (t, \rho, \varphi, z)$$

$$\Pi = \phi dt \wedge d\rho + \psi \rho d\phi \wedge dz$$

Equations of Motion?

$$\Box\Pi = (\Box\phi + \frac{\phi}{\rho^2}) dt \wedge d\rho - \partial_\varphi \frac{2\phi}{\rho} dt \wedge d\varphi$$

$$+(\Box\psi + \frac{\psi}{\rho^2}) * (dt \wedge d\rho) - \partial_\varphi \frac{2\psi}{\rho} * (dt \wedge d\varphi)$$

TE Modes in Cylindrical Coordinates

Define

$$\Pi_A = \phi_A dt \wedge dz + \psi_A * (dt \wedge dz),$$

$$\Pi_R = \phi_R dt \wedge d\rho + \psi_R * (dt \wedge d\rho).$$

We start with

$$A = \delta \Pi_A = \delta \Pi_R - G. \qquad (20)$$

$$B_z = B_{k\omega} \sin(kz) g(\rho, \varphi) e^{-i\omega t},$$

hence

$$\phi_A = 0, \psi_A = \frac{-B_{k\omega}}{\omega^2 - k^2} \sin(kz) g(\rho, \varphi) e^{-i\omega t},$$

TE Modes in Cylindrical Coordinates

From (20) we obtain

Radial Modes

$$\phi_R = \frac{iB_{k\omega}}{\rho\omega(\omega^2 - k^2)} \sin(kz)\partial_\varphi g(\rho, \varphi)e^{-i\omega t}$$

$$\psi_R = \frac{-B_{k\omega}}{k(\omega^2 - k^2)} \cos(kz)\partial_\rho g(\rho, \varphi)e^{-i\omega t}$$

$$G_t = \frac{iB_{k\omega}}{\rho\omega(\omega^2 - k^2)} \sin(kz)\partial_\rho\partial_\varphi g(\rho, \varphi)e^{-i\omega t}, G_\rho = 0$$

$$G_z = \frac{B_{k\omega}}{k\rho(\omega^2 - k^2)} \cos(kz)\partial_\rho\partial_\varphi g(\rho, \varphi)e^{-i\omega t}, G_\varphi = 0$$

Azimuthal Cylindrical Coordinates

Define

$$\Pi_P = \phi_P dt \wedge d\varphi + \psi_P * (d \wedge d\varphi),$$

and again start with

$$\delta \Pi_A = \delta \Pi_P - G. \tag{21}$$

This yields

Azimuthal Modes

$$\phi_P = \frac{-iB_{k\omega}}{\omega(\omega^2 - k^2)} \sin(kz) \rho \partial_\rho g(\rho, \varphi) e^{-i\omega t}$$

$$\psi_P = \frac{B_{k\omega}}{\rho k(\omega^2 - k^2)} \cos(kz) \partial_\varphi g(\rho, \varphi) e^{i - \omega t}$$

$$G_t = -\frac{1}{\rho^2} \partial_\varphi \phi_P, G_\rho = 0$$

$$G_z = \frac{1}{\rho} \partial_\rho \psi_P, G_\varphi = 0$$

Ongoing and Future Work

1. Determine the equations of motion for the radial and azimuthal cylindrical cases.
2. Consider the polar and azimuthal spherical cases.
3. Examine the boundary conditions of all of the presented cases.
4. Consider geometries with non-trivial topology.