Potentials with compact support

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Outline

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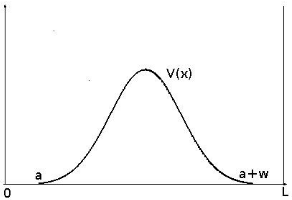
Pistons with Compact Supported Potentials

Consider a piston



with Dirichlet boundary conditions and middle plate represented by $V(x)$

Zeta Function

- State the problem as an eigenvalue equation

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- Find the Casimir energy and functional determinant
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  - Evaluate $\zeta'(0)$ and $\zeta\left(-\frac{1}{2}\right)$

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Eigenvalue Problem

We want to solve the eigenvalue equation

$$\left(-\frac{\partial^2}{\partial x^2} + V(x)\right)\phi(x) = \lambda^2\phi(x)$$

in $[0, L]$ with Dirichlet boundary conditions.

Eigenvalue Problem

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in $[0, L]$ with Dirichlet boundary conditions. Let $V(x)$ be supported in $[a, a+w]$ such that $w > 0$ and

$$\int_a^{a+w} V(x)dx = \sigma$$

is fixed.

Assumptions on ϕ

- Continuous

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- Incommensurable lengths ($a/L \notin \mathbb{Q}$)
- Normalization ($\phi(a) = \phi(a+w) = 1$)

Secular Condition

Integrating the equation around the support gives

$$\lambda^2 \int_{a-\epsilon}^{a+w+\epsilon} \phi(x)dx = - \int_{a-\epsilon}^{a+w+\epsilon} \frac{\partial^2 \phi(x)}{\partial x^2} dx + \int_{a-\epsilon}^{a+w+\epsilon} V(x)\phi(x)dx$$

Letting $\epsilon \to 0$ and using the solutions of the differential equation, lead to

$$\lambda \cot(\lambda a) + \lambda \cot(\lambda(L - w - a)) + \sigma - \lambda^2 w = \int_a^{a+w} \phi'(x) \left[\int_a^x \left(V(u) - \lambda^2 \right) du \right] dx$$

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Characteristic Function

In order to remove singularities at the origin, define

$$F(\lambda) = -\frac{1}{\lambda}\sin(\lambda(L-w)) - \frac{\sigma}{\lambda^2}\sin(\lambda a)\sin(\lambda(L-w-a))$$
$$+w\sin(\lambda a)\sin(\lambda(L-w-a))$$
$$+\frac{\sin(\lambda a)\sin(\lambda(L-w-a))}{\lambda^2}\int_a^{a+w}\phi'(x)\left[\int_a^x\left(V(u)-\lambda^2\right)du\right]dx$$

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Asymptotics for large y

In order to write the zeta function as a contour integral and deform it to the imaginary axis, we set $\lambda = iy$, and when $y \to \infty$,

$$\frac{\partial^2 \phi(x)}{\partial x^2} + V(x)\phi(x) + y\phi(x) \sim \frac{\partial^2 \phi(x)}{\partial x^2} + y\phi(x)$$

so that

$$\int_a^{a+w} \phi(x)dx \sim c_V$$

where c_V is a constant depending on $V(x)$ and independent of a

Thus,

$$F(iy) \sim \left(w - \left(\frac{1}{2y} + \frac{d_V}{4y^2} \right) \right) e^{y(L-w)}$$

where

$$d_V = \sigma + c_V$$

depends only on the potential $V(x)$ and is independent of a.

Therefore, we have the asymptotics of the logarithm

$$\ln F(iy) \sim y(L-w) + \ln w - \sum_{k=1}^{\infty} y^{-k} \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-j}{j} 2^{-k} w^{j-k} d_V^j$$

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Remark

In the limit when $w = 0$ the asymptotic expansion contains a logarithmic term in y recovering the semitransparent result

$[0, L] \times \mathcal{N}$ setting

Considering the problem in $[0, L] \times \mathcal{N}$, where $\mathcal{N}$ is a $(D-1)$-dimensional compact Riemannian manifold possibly with boundary, we have that

$$\zeta(s) = \sum_{\lambda,\ell} \left(\lambda^2 + \eta_\ell^2\right)^{-s} \qquad \text{for } \Re(s) > \frac{D}{2}$$

where η_ℓ is the spectrum in $\mathcal{N}$

Therefore

$$\zeta(s) = \sum_{\ell} \frac{\sin \pi s}{\pi} \int_{\eta_\ell}^{\infty} dy \ \left(y^2 - \eta_\ell^2\right)^{-s}$$
$$\times \frac{d}{dy}\left\{ \ln F(iy) - y(L-w) - \sum_{k=0}^{N} \frac{d_k}{y^k} \right\}$$
$$- \frac{1}{2\Gamma(s)} \left\{ (L-w) \frac{\Gamma\left(s-\frac{1}{2}\right)}{\sqrt{\pi}} \zeta_{\mathcal{N}}\left(s - \frac{1}{2}\right) \right.$$
$$\left. + \sum_{k=1}^{N} k d_k \frac{\Gamma\left(s+\frac{k}{2}\right)}{\Gamma\left(1+\frac{k}{2}\right)} \zeta_{\mathcal{N}}\left(s+\frac{k}{2}\right) \right\} \quad \text{for } \Re(s) > \frac{D-N-2}{2}$$

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where

$$d_0 = d_V \quad \text{and} \quad d_k = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-j}{j} 2^{-k} w^{j-k} d_V^j$$

Functional Determinant and Casimir Energy

Letting $N = D - 1$ we can calculate $\zeta'(0)$ which gives the functional determinant

$$\exp\left(\zeta'(0)\right)$$

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Letting $N = D - 1$ we can calculate $\zeta'(0)$ which gives the functional determinant

$$\exp\left(\zeta'(0)\right)$$

and setting $N = D$ leads to the Casimir energy

$$\zeta\left(-\frac{1}{2}\right)$$

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Casimir Force

In order to calculate the Casimir Force, we can compute

$$-\frac{1}{2}\frac{\partial}{\partial a}\zeta\left(-\frac{1}{2}\right)$$

Attractive or repulsive?

- For some specific potentials, the force is attractive to the closest wall (delta, rectangular)

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- For some specific potentials, the force is attractive to the closest wall (delta, rectangular)
- Attractive for all potentials? (It seems to be the case)

Thank You!

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