WKB approximation of a Power Wall

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Introduction

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2. We find action along different paths
3. We look at the amplitude

A Quantum Particle

Consider a quantum particle subject to a bounded potential V(x,t). The wavefunction of the particle can be written as

$$\psi(x, t) = A(x, t)e^{\frac{i}{\hbar}S(x,t)}$$

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where A(x,t) and S(x,t) are the amplitude and the action. Substituting this into the time-dependent Schrödinger equation, we get,

$$A[\frac{\partial S}{\partial t}+\frac{1}{2m}(\nabla S)^2+V]-i\hbar[\frac{\partial A}{\partial t}+\frac{1}{m}(\nabla A\cdot\nabla S)+\frac{1}{2m}A\Delta S]-\frac{\hbar^2}{2m}\Delta A=0$$

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If we take $\hbar \to 0$,

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V(x,t) = 0.$$

This is the Hamilton-Jacobi equation. S(x,t) is interpreted as classical action.

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Now if we take total time derivative of the action, we get,

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Conversely, define $S(x, y, t) = \int_0^t L(x(u), \dot{x}(u))du + S_0$. This also solves Hamilton-Jacobi.

Construction of action S

Case I

In this case, we look into a system where x(t) lies on the left side of origin.

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$q(\tau) = A\cos(\omega\tau) + B\sin(\omega\tau)$
$v_1 = -\frac{y}{t_1}; v_2 = \frac{x}{t-t_2}$
$q(t_1) = 0; q(t_2) = 0$
$\dot{q}(t_1) = v_1; \dot{q}(t_2) = v_2$

Calculating S

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$S = \int_0^t Ldt = \int_0^{t_1} Ldt + \int_{t_1}^{t_2} Ldt + \int_{t_2}^{t} Ldt$

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For path 2, $L = \frac{1}{4}[\dot{q}^2 - \omega^2 q^2]$

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Result:
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This constant A agrees with what Fernando Mera was talking about in earlier presentation.

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So, $0 < \tilde{\Omega} < \pi$

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