Scaling the Power Wall

This research is in collaboration with **Jef Wagner** (VAP, Spring '10), with some participation by my RAs (on NSF PHY-0554849):

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Vacuum energy

Real physics (Casimir effect)

Two neutral good conductors tend to attract each other at mesoscopic separation. Although its physical mechanism is interaction between quantum fluctuations of charges in the conductors, the effect can be calculated (for *perfect* conductors) from the energy in the electromagnetic field in the empty space between them.

This is an arena for some nice spectral theory!

A scalar field

$$\frac{\partial^2\phi}{\partial {x_0}^2} = \nabla^2\phi\,, \quad \text{Dirichlet B.C.}$$

$$T^{00} = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x_0}\right)^2 - \phi\nabla^2\phi\right] \quad \text{(energy density).}$$

$$T^{jj} = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x_j}\right)^2 - \phi\frac{\partial^2\phi}{\partial {x_j}^2}\right] \quad \text{(pressure).}$$

(Curvature coupling constant $\xi = \frac{1}{4}$.)

A flat, perfectly reflecting wall at $z = 0$

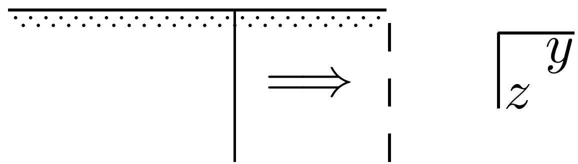
After removal of the zero-point energy of individual normal modes,

$$\langle T^{00} \rangle \propto z^{-4}, \qquad \langle T^{zz} \rangle \propto 0, \qquad \langle T^{yy} \rangle \propto -z^{-4}$$

(expectation values in the ground state). So

- The total energy (per unit area), $E = \int \langle T^{00} \rangle \, dz$, is infinite.
- The force on the wall is 0, because moving it does not change the energy.

- The force (per unit length) on a perpendicular wall, $\int \langle T^{yy} \rangle \, dz$, would be $-E$. That is a correct surface tension relation! The total energy grows linearly with surface area. The work done by the force is the negative of the change in the system's own energy.



In general, $\dfrac{\partial E_{\text{tot}}}{\partial h} = -\displaystyle\int_S p_h$ for a general parameter h.

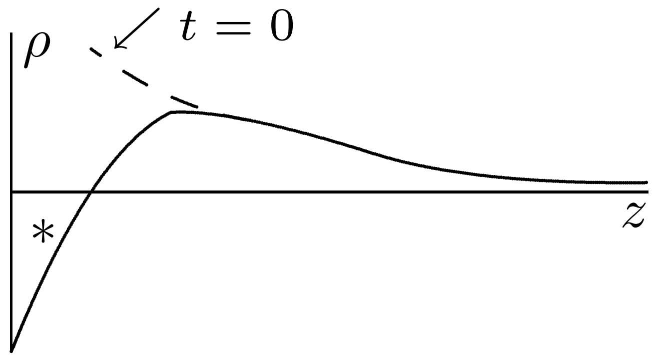
Can we get rid of the infinities?

A real material is not perfectly conducting to arbitrarily high frequencies.

1. Model the material seriously. (difficult condensed-matter physics, no longer nice spectral theory)
2. Insert an ultraviolet cutoff, $e^{-\omega t}$, in the mode sums.

   This time $\dfrac{\partial E_{\text{tot}}}{\partial h} = +2 \displaystyle\int_S p_h$!

The factor 2 (which should be -1) is $+(n-1)$ in dimension n.



The energy-balance equation is disrupted because the dip $*$ at the boundary occurs only in the (mollified) energy density, not in the pressure.

(These observations show one of the virtues of examining local quantities, not just total energy.)

The same thing (with same bad factor -2) happens for energy density and pressure on a sphere.
[M. Schaden (summer '09 visitor); Zhonghai (Bruce) Liu (Ph.D. '09)]

3. A steeply rising smooth potential ("soft wall")
   - mocks up a reflecting wall.
   - should define a nonsingular, internally consistent theory.

In principle this can be done for a spherical (or general) boundary, but for now we study only a plane one.

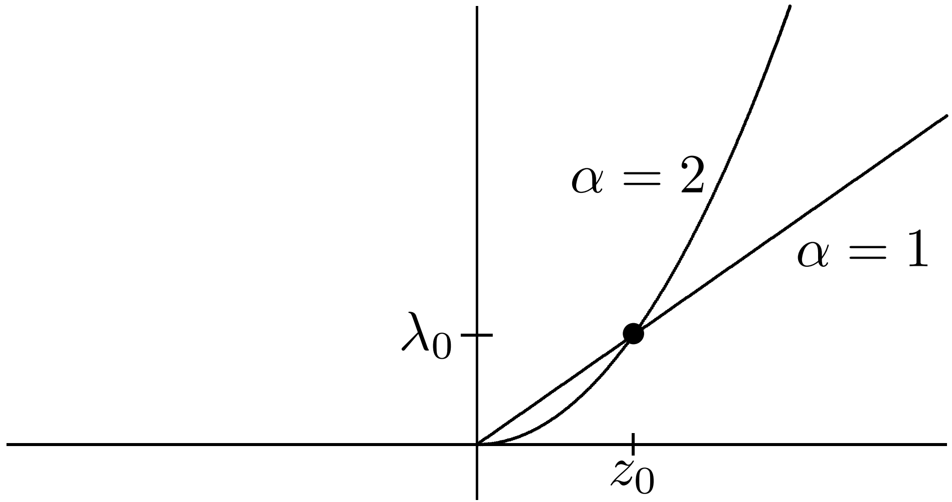
The model

$$v(x,y,z) = \begin{cases} 0, & z < 0 \\ \lambda z^{\alpha}, & z > 0. \quad (1 \le \alpha \in \mathbf{R}) \end{cases}$$

Get dimensions right: $v = \lambda_0 \left(\dfrac{z}{z_0} \right)^{\alpha}$.

Only one length scale: $\hat{z} = \lambda^{\frac{-1}{\alpha+2}} = \left(\dfrac{{z_0}^{\alpha}}{\lambda_0} \right)^{\frac{1}{\alpha+2}}$.

For any α, $v(z_0) = \lambda_0$; increasingly steep as $\alpha \to \infty$.



$$\frac{\partial^2 \phi}{\partial {x_0}^2} = \nabla^2 \phi - v(z)\phi.$$

$$\varphi(\mathbf{r}, x^0) = \sum_n \frac{1}{\sqrt{2\omega_n}} \left[a_n \phi_n(\mathbf{r}) e^{-i\omega_n x^0} + a_n^\dagger \phi_n(\mathbf{r}) e^{i\omega_n x^0}\right].$$

$$T^{00} = \lim_{t\to 0} -\frac{1}{2} \frac{\partial^2 \overline{T}}{\partial t^2}, \quad \text{etc.} \qquad (t \leftrightarrow ix^0)$$

$$\overline{T}(t, \mathbf{r}, \mathbf{r}') = \sum_n \frac{\phi_n(\mathbf{r})\phi_n(\mathbf{r}')}{-\omega_n} e^{-\omega_n t}.$$

$$\left(\frac{\partial^2}{\partial t^2} + \nabla^2 - v(z)\right)\overline{T}(t, x, y, z, z') = 2\delta(t)\delta(x)\delta(y)\delta(z - z').$$

Eigenfunctions

$$\phi_n(\mathbf{r}) = e^{i\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \phi_p(z)$$

$$(\mathbf{r}_\perp,\ \mathbf{k}_\perp \in \mathbf{R}^2, \quad z \in \mathbf{R},\ p \in (0, \infty).)$$

$$\left(-\frac{\partial^2}{\partial z^2} + v(z) - p^2 \right) \phi_p(z) = 0.$$

$$\phi_p(z) = \sqrt{\frac{2}{\pi}} \sin[pz - \delta(p)] \quad \text{when } z < 0.$$

When $z > 0$, $\quad \phi_p(z) = C(p) P_\alpha\left(\frac{z}{\hat{z}}, (\hat{z}p)^2\right)$,

$$\left(-\frac{d^2}{dz^2} + z^\alpha - E\right) P_\alpha(z, E) = 0, \quad P_\alpha(+\infty, E) = 0.$$

$$P_1(z, E) \propto \mathrm{Ai}(z - E), \qquad P_2(z, E) \propto D_{\frac{1}{2}(E-1)}(\sqrt{2}\, z).$$

For hard wall at z_0, $P_\infty(z, E) \propto \sin[\sqrt{E}(z - z_0)]$.
(Henceforth usually $\hat{z} = 1$, $\sqrt{E} = p$ $(z_0 = 1 = \lambda_0)$.)

The solutions must match at $z = 0$:

$$\tan\big(\delta(p)\big) = -p\,\frac{P_\alpha(0,p^2)}{P'_\alpha(0,p^2)}\,.$$

$$C(p)^2 = \frac{2}{\pi}\,\frac{1}{P_\alpha(0,p^2)^2 + p^{-2}P'_\alpha(0,p^2)^2}\,.$$

Even for $P = \mathrm{Ai}$, these formulas are unpleasant.

Small p

When $p = 0$ the solution is known:

$$P_\alpha(z,0) = z^{1/2} K_{\frac{1}{\alpha+2}}\left(\frac{2}{\alpha+2}\, z^{\frac{\alpha+2}{2}}\right).$$

Perturbation expansion:

$$P_\alpha(z,E) = P_\alpha(z,0) + E P_\alpha^{(1)}(z) + \cdots.$$

$$\delta(p) = p(\alpha+2)^{\frac{2}{\alpha+2}}\, \Gamma\!\left(\frac{\alpha+3}{\alpha+2}\right) \Gamma\!\left(\frac{\alpha+1}{\alpha+2}\right)^{-1} + O(p^3).$$

Large p (WKB)

$$\phi_p(z) \sim [p^2 - v(z)]^{-\frac{1}{4}} \cos\left[\int_z^a \sqrt{p^2 - v(\tilde{z})}\, d\tilde{z} - \frac{\pi}{4}\right],$$

$$\text{turning point } a = p^{2/\alpha}.$$

$$\delta(p) = \int_0^a \sqrt{p^2 - v(z)}\, dz + \frac{\pi}{4} \mod \pi$$

$$= \frac{1}{\alpha}\, p^{1+2/\alpha} \mathrm{B}\left(\frac{3}{2}, \frac{1}{\alpha}\right) + \frac{\pi}{4}\,.$$

$\alpha = 1$ (Airy function):

$$\delta(p) \sim \begin{cases} p\,3^{2/3}\Gamma(\frac{4}{3})/\Gamma(\frac{2}{3}), & p \to 0, \\ \dfrac{2p^3}{3} + \dfrac{\pi}{4}\,, & p \to \infty. \end{cases}$$

$\alpha = 2$ (parabolic cylinder function):

$$\delta(p) \sim \begin{cases} 2p\,\Gamma(\frac{5}{4})/\Gamma(\frac{3}{4}), & p \to 0, \\ \dfrac{\pi p^2}{4} + \dfrac{\pi}{4}\,, & p \to \infty. \end{cases}$$

Cylinder kernel calculations

Recall

$$\overline{T}(t,\mathbf{r},\mathbf{r}') = \sum_n \frac{\phi_n(\mathbf{r})\phi_n(\mathbf{r}')}{-\omega_n}\, e^{-\omega_n t}.$$

$$\left(\frac{\partial^2}{\partial t^2} + \nabla^2 - v(z)\right)\overline{T}(t,\mathbf{r}_\perp,z,z') = 2\delta(t)\delta^{(2)}(\mathbf{r}_\perp)\delta(z-z').$$

$$\hat{T}(\omega,\mathbf{k}_\perp,p) = \frac{-2}{(2\pi)^{3/2}}\,\frac{\phi_p(z')}{\omega^2 + {k_\perp}^2 + p^2}\,.$$

Cartesian calculations

$$\overline{T}(t, \mathbf{r}_\perp, z, z') = -\frac{1}{2\pi}\int_0^\infty dp\, Y(s,p)\phi_p(z)\phi_p(z'),$$

$$Y(s,p) \equiv \frac{e^{-sp}}{s}, \qquad s \equiv \sqrt{t^2 + |\mathbf{r}_\perp|^2}\,.$$

In potential-free region, $z < 0$,

$$\overline{T} = -\frac{1}{\pi^2}\int_0^\infty dp\, Y(s,p)\sin\bigl(pz - \delta(p)\bigr)\sin\bigl(pz' - \delta(p)\bigr).$$

$$
\begin{aligned}
\overline{T} =& -\frac{1}{2\pi^2}\,\frac{1}{t^2 + {r_\perp}^2 + (z - z')^2} \\
&+ \frac{1}{2\pi^2}\int_0^\infty dp\, Y(s,p)\cos\bigl(p(z+z') - 2\delta(p)\bigr) \\
&\equiv \overline{T}_{\rm free} + \overline{T}_{\rm ren}\,.
\end{aligned}
$$

Hard wall: $\delta(p) = z_0 p \;\Rightarrow$ (correctly)

$$
\overline{T}_{\rm ren} = \frac{1}{2\pi^2}\,\frac{1}{t^2 + {r_\perp}^2 + (z + z' - 2z_0)^2}\,.
$$

The bad news:

$$\overline{T}_{\mathrm{ren}} = \frac{1}{2\pi^2} \int_0^\infty dp\, \frac{e^{-sp}}{s}\, \cos\bigl(p(z+z') - 2\delta(p)\bigr)$$

is poorly convergent when $s \equiv \sqrt{t^2 + {r_\perp}^2}$ is small, which is precisely where we want it. In fact, we should be able to take $s = 0$ and get a finite answer when $z + z' > 0$, but the integrand is pointwise infinite there!

There is a genuine divergence for $\delta(p) = Ap + B$ unless $B = 0$. Asymptotics for *small* p as well as large are critical. Presumably large $\mathbf{k}_\perp$ is at fault, so . . .

Polar calculations

Do the integral in polar coordinates in the Fourier space: $(Z \equiv z + z'$, $\mathbf{s} = (t, \mathbf{r}_\perp)$, $\mathbf{v} = (\omega, \mathbf{k}_\perp))$

$$\overline{T}_{\text{ren}} = \frac{1}{4\pi^4} \int_0^\infty dp \int_{\mathbf{R}^3} d\mathbf{v}\, \frac{e^{i\mathbf{v}\cdot\mathbf{s}}}{v^2 + p^2}\, \cos\bigl(pZ - 2\delta(p)\bigr)$$

$$= \frac{1}{\pi^3} \int_0^\infty d\rho \int_0^1 du\, s^{-1} \sin(s\rho\sqrt{1-u^2})$$

$$\times \cos\bigl(Z\rho u - 2\delta(\rho u)\bigr).$$

With $s = 0$ and $z = z'$ (should give $\langle\phi(z)^2\rangle$),

$$\overline{T}_{\rm ren}(0,0,z,z) = \frac{1}{\pi^3}\int_0^\infty d\rho \int_0^1 du\,\rho\sqrt{1-u^2} \times \cos\bigl(2z\rho u - 2\delta(\rho u)\bigr).$$

Recall $\delta(p) = \tan^{-1}\dfrac{-p\mathrm{Ai}(-p^2)}{\mathrm{Ai}'(-p^2)}$ for $\alpha = 1$, for example. Padé interpolation between small and large p asymptotics can be used to approximate it.

Numerics (BARELY STARTED)

- The oscillatory integrals make *Mathematica* scream in pain, even with Riesz–Cesàro averaging.
- By eye, the expected $(z-1)^{-2}$ behavior is emerging.
- The inner integral seems to approach a nonzero constant as $\rho \to \infty$. (The Padé approximation is too crude?)
- The problem does not quite fit the framework of neo-Filon quadrature (Iserles, Nørsett, S. Olver).