The Casimir Effect for Generalized Piston Geometries

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The Generalized Piston Geometry

Let $\mathscr{N}$ be a smooth, compact Riemannian d-dimensional base manifold, $\mathcal{I} = [a, b] \subset \mathbb{R}$, and $f(r) \in C^{\infty}(\mathscr{M})$ with $f(r) > 0$ be a *warping function.* The generalized piston is defined as the $D = d + 1$ dimensional compact manifold $\mathscr{M} = \mathcal{I} \times_f \mathscr{N}$ locally described by the line element

$$\mathrm{d}s^2 = \mathrm{d}r^2 + f^2(r)\mathrm{d}\Sigma^2_{\mathscr{N}} \quad , \quad r \in \mathcal{I} \, .$$

Piston Configuration

- $\mathscr{N}_R$ is a cross section of $\mathscr{M}$ at $r = R \in (a, b)$.
- $\mathscr{N}_R$ naturally divides $\mathscr{M}$ in two regions
  - $M_I = [a, R] \times \mathscr{N}$, with $\partial M_I = \mathscr{N}_a \cup \mathscr{N}_R$,
  - $M_{II} = (R, b] \times \mathscr{N}$, with $\partial M_{II} = \mathscr{N}_R \cup \mathscr{N}_b$,
- The piston configuration is $M_I \cup_{\mathscr{N}_R} M_{II}$, where the piston itself is modelled by the cross section $\mathscr{N}_a$.

Remarks:

- M_I and M_{II} have *different geometry* unlike standard Casimir pistons.
- By setting $f(r) = r$ one recovers the conical piston.

A 2-Dimensional Example: S^1 as Base Manifold

Let $g(r)$ be the warping function with $r \in [0, a]$ and let $\mathscr{N} = S^1$. By parametrizing the surface as

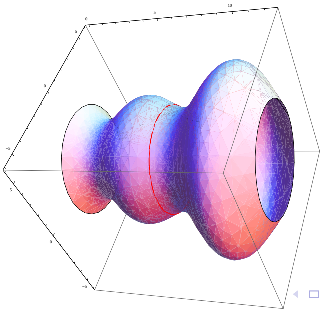
$$\Phi(r, \phi) = (f^{-1}(r) \cos\phi, f^{-1}(r) \sin\phi, g(f^{-1}(r)))$$

with $0 \le \phi < 2\pi$ and

$$f(u) = \int_0^u \sqrt{1 + g'^2(\nu)} \mathrm{d}\nu \ , \quad 0 < u \le a \ ,$$

the line element becomes

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \left(f^{-1}(r)\right)^2 \mathrm{d}\phi^2 \quad ,$$



Let $\varphi_p \in \mathcal{L}^2(\mathscr{M})$ with $p = (I, II)$, we consider the eigenvalue equation

$$-\Delta_{\mathscr{M}} \varphi_p = \alpha_p^2 \varphi_p \ .$$

By using separation of variables we represent the eigenfunctions as $\varphi_p(r, X) = u_{\alpha_p}(r)\Phi_p(X)$ where

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + d\frac{f'(r)}{f(r)}\frac{\mathrm{d}}{\mathrm{d}r} + \alpha_p^2 - \frac{\nu^2}{f^2(r)}\right) u_{\alpha_p}(r) = 0 \ .$$

and

$$-\Delta_{\mathscr{N}} \Phi_p(X) = \nu^2 \Phi_p(X) \ .$$

The spectral zeta function associated with the generalized piston can be written as

$$\zeta(s) = \zeta_I(s) + \zeta_{II}(s) \ , \quad \text{where} \quad \zeta_p(s) = \sum_{\alpha_p} \alpha_p^{-2s} \ .$$

In the framework of zeta function regularization the Casimir energy is

$$E_{\text{Cas}}(R) = \lim_{\varepsilon \to 0} \frac{\mu^{2\varepsilon}}{2} \zeta_M \left(\varepsilon - \frac{1}{2}, R \right) .$$

In the limit $\varepsilon \to 0$, one finds the expression for the energy

$$E_{\text{Cas}}(R) = \frac{1}{2} \text{FP} \zeta \left(-\frac{1}{2}, R \right) + \frac{1}{2} \left(\frac{1}{\varepsilon} + \ln \mu^2 \right) \text{Res}\, \zeta \left(-\frac{1}{2}, R \right) + O(\varepsilon) ,$$

while the corresponding force on the piston is

$$F_{\text{Cas}}(R) = -\frac{\partial}{\partial R} E_{\text{Cas}}(R) .$$

Remark: An unambiguous prediction of the force can be obtained only if $\frac{\partial}{\partial R} \text{Res}\, \zeta \left(-\frac{1}{2}, R \right) = 0$.

Spectral Zeta Function

An implicit equation for the eigenvalues α_p in region I and II is obtained by imposing boundary conditions. For Dirichlet BC's we set

$$u_{\alpha_I}(a,\nu) = u_{\alpha_I}(R,\nu) = 0\ , \quad \text{and} \quad u_{\alpha_{II}}(R,\nu) = u_{\alpha_{II}}(b,\nu) = 0\ .$$

The spectral zeta function for the piston can be written as

$$\zeta(s) = \sum_{p\in\{I,II\}} \sum_{\nu} d(\nu)\zeta_p^{\nu}(s)\ ,$$

where, by using Cauchy residue theorem, $\zeta_p^{\nu}(s)$ has the following integral representation (with $x_I = R$ and $x_{II} = b$)

$$\zeta_p^{\nu}(s) = \frac{\sin \pi s}{\pi} \int_{\frac{m}{\nu}}^{\infty} dz (\nu^2 z^2 - m^2)^{-s} \frac{\partial}{\partial z} \ln u_{i\nu z}(x_p,\nu)\ .$$

Remarks:

- The above integral representation is valid for $1/2 < \Re(s) < 1$ and, hence, the analytic continuation to the region $\Re(s) \leq 1/2$ needs to be performed.
- For a general warping function $f(r)$ the eigenfunctions u_{α_p} are *not* known explicitly!

Asymptotic Expansion of the Eigenfunctions

For the analytic continuation of $\zeta(s)$ the explicit knowledge of the eigenfunctions is not necessary. *We only need their uniform asymptotic expansion.* Let us consider the following *ansatz* for the eigenfunctions of the radial equation

$$u_{i\nu z}(r,\nu) = f^{-d}(r)\Psi_\nu(z,r) \; .$$

The function $\Psi_\nu(z,r)$ satisfies the equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + q(\nu,z,r)\right)\Psi_\nu(z,r) = 0 \; ,$$

with

$$q(\nu,z,r) = -\nu^2\left(z^2 + \frac{1}{f^2(r)}\right) - \frac{d}{2}\frac{f''(r)}{f(r)} - \frac{d(d-2)}{4}\frac{f'^2(r)}{f^2(r)} \; .$$

To find the asymptotic expansion of Ψ and, in turn, of u for $\nu \to \infty$ we utilize the *WKB method*. We introduce the function

$$\mathcal{S}(\nu,z,r) = \frac{\partial}{\partial r}\ln\Psi_\nu(z,r) \; ,$$

which satisfies the non-linear differential equation

$$\mathcal{S}'(\nu,z,r) = -q(\nu,z,r) - \mathcal{S}^2(\nu,z,r) \; .$$

We consider the following form for asymptotic expansion of the function $\mathcal{S}$

$$\mathcal{S}(\nu, z, r) \sim \nu\, S_{-1}(z, r) + S_0(z, r) + \sum_{i=1}^{\infty} \frac{S_i(z, r)}{\nu^i} \;.$$

The terms of the expansion satisfy the recursion relation for $i \geq 1$

$$S^{\pm}_{i+1}(z, r) = -\frac{1}{2S^{\pm}_{-1}(z, r)} \left[S'^{\pm}_{i}(z, r) + \sum_{n=0}^{i} S^{\pm}_{n}(z, r) S^{\pm}_{i-n}(z, r) \right] \;,$$

with

$$S^{\pm}_{-1}(z, r) = \pm\sqrt{z^2 + \frac{1}{f^2(r)}} \;, \qquad S^{\pm}_{0}(z, r) = -\frac{1}{2}\frac{\partial}{\partial r} \ln S^{\pm}_{-1}(z, r) \;,$$

$$S^{\pm}_{1}(z, r) = -\frac{1}{2S^{\pm}_{-1}(z, r)} \left[-\frac{d}{2}\frac{f''(r)}{f(r)} - \frac{d(d-2)}{4}\frac{f'^2(r)}{f^2(r)} + S^2_0(z, r) + S'_0(z, r) \right] \;.$$

Asymptotic Expansion of the Eigenfunctions

The large-ν asymptotic expansion of the eigenfunctions $u_{i\nu z}$ is then given by

$$u_{i\nu z}(r,\nu) = f^{-d}(r)\left[A\exp\left\{\int_a^r \mathcal{S}^{+}(\nu,z,t)\mathrm{d}t\right\} + B\exp\left\{\int_a^r \mathcal{S}^{-}(\nu,z,t)\mathrm{d}t\right\}\right] .$$

By imposing Dirichlet boundary conditions in region I we obtain

$$\begin{aligned} \ln u_{i\nu z}(R,\nu) &= -\ln 2\nu - \frac{1}{2}\ln\left(z^2 + \frac{1}{f^2(a)}\right) + \frac{1}{4}\ln\left[\frac{1+z^2 f^2(a)}{1+z^2 f^2(R)}\right] \\ &+ \frac{d-1}{2}\ln\frac{f(a)}{f(R)} + \nu\int_a^R S_{-1}^{+}(z,t)\mathrm{d}t + \sum_{i=1}^{\infty}\frac{\mathcal{M}_i(z,a,R)}{\nu^i} . \end{aligned}$$

Remark:

- The uniform asymptotic expansion for the eigenfunctions in region II is obtained from the above with the replacement $a \to R$ and $R \to b$.

Analytic Continuation of the Spectral Zeta Function

From the integral representation of $\zeta(s)$ we add and subtract L leading terms of the asymptotic expansion to obtain, in region I,

$$\zeta_I(s) = Z_I(s) + \sum_{i=-1}^{L} A_i^{(I)}(s) \ ,$$

with $Z_I(s)$ analytic for $\Re s > (d - L - 1)/2$. By defining $\zeta_{\mathcal{N}}(s) = \sum_\nu \nu^{-2s}$ we find

$$A_{-1}^{(I)}(s) = \frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \zeta_{\mathcal{N}} \left(s - \frac{1}{2}\right) \int_a^R f^{2s-1}(t) \mathrm{d}t \ ,$$

$$A_0^{(I)}(s) = -\frac{1}{4} \zeta_{\mathcal{N}}(s) \left[f^{2s}(a) + f^{2s}(R)\right] \ ,$$

$$A_i^{(I)}(s) = -\frac{1}{\Gamma(s)} \zeta_{\mathcal{N}} \left(s + \frac{i}{2}\right) \Omega_i(s, a, R) \ , \quad i \geq 1 \ .$$

Remarks:

- Once again similar results are obtained in region II once the replacement $a \to R$ and $R \to b$ is performed.
- The spectral zeta function on M depends *explicitly* on the spectral zeta function on $\mathcal{N}$.

The Casimir Force on the Piston

The Casimir energy for the generalized piston is obtained as

$$E_{\mathrm{Cas}}(R) = -\frac{1}{2}\lim_{\varepsilon\to 0}\left[\zeta_I\left(\varepsilon-\frac{1}{2},R\right)+\zeta_{II}\left(\varepsilon-\frac{1}{2},R\right)\right] \,,$$

and the corresponding force on the piston has the form

$$\begin{aligned} F_{\mathrm{Cas}}(R) &= -\frac{1}{2}Z_I'\left(-\frac{1}{2},R\right)-\frac{1}{2}Z_I'\left(-\frac{1}{2},R\right) \\ &+\sum_{n=1}^{[D/2]}\left[\mathrm{FP}\zeta_{\mathscr{N}}\left(n-\frac{1}{2}\right)\mathcal{A}(R)+\mathrm{Res}\zeta_{\mathscr{N}}\left(n-\frac{1}{2}\right)\mathcal{B}(R)\right] \\ &-\left(\frac{1}{\varepsilon}+\ln\mu^2\right)\left[\mathrm{Res}\zeta_{\mathscr{N}}\left(n-\frac{1}{2}\right)\frac{f'(R)}{f^2(R)}-\sum_{n=1}^{[D/2]}\mathrm{Res}\zeta_{\mathscr{N}}\left(n-\frac{1}{2}\right)\mathcal{C}(R)\right] \,. \end{aligned}$$

Remarks:

- The Casimir force is divergence-free if $\dim\mathscr{N} = 2k$ and $\partial\mathscr{N} = \emptyset$.
- $\mathcal{A}(R)$, $\mathcal{B}(R)$, and $\mathcal{C}(R)$ depend on $f^{(n)}(R)$, $n \geq 1$. The Casimir force is *always* unambiguous when $f(r)$ is *constant* (i.e. a generalized cylinder).

Concluding Remarks

- The behavior of the Casimir force as a function of the position of the piston can be studied (at least numerically) once a warping function and a base manifold have been specified.
- The formalism can be modified in order to study the generalized piston configuration when Neumann or Hybrid boundary conditions are imposed.
- It would be interesting to consider a modification of the warped product geometry to include the *warped torus*, a compact manifold $\mathcal{T} = S^1 \times_f \mathcal{N}$ with and the *periodic condition* $f(0) = f(2\pi)$. One could study the Casimir force between two cross-sections of the warped torus (generalization of the annular pistons).

References

- (with K. Kirsten)
  The Casimir Effect for Generalized Piston Geometries,
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