Calculation of Highly Oscillatory Integrals by Quadrature Methods

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Outline

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1. Motivation
   - Study of Vacuum Energy
   - Oscillatory Integrals
   - Earlier Literature

2. Our Results
   - Main Results
   - Implementation

A model for Vacuum Energy

Our model of quantum vacuum energy density near the boundary has the form λz^α.

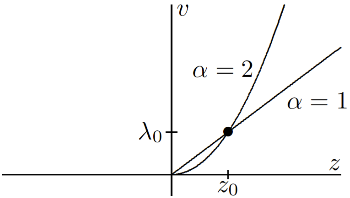

Figure: Steeply rising potential near the boundary

Why are we interested?

Spectral analysis of the rising potential gives Energy momentum tensor:

$$\overline{T}(z) = \frac{1}{\pi^3} \int_0^\infty d\rho \int_0^1 du \sqrt{1-u^2} \cos(2z\rho u - 2\delta(\rho u)) \qquad (1)$$

where,

$$\delta(u) = \mathrm{ArcTan}\Big(-u \Big(\frac{\mathrm{AiryAi}(-u^2)}{\mathrm{AiryAi}'(-u^2)} \Big) \Big)$$

What does it look like?

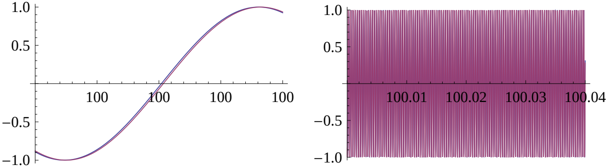

Figure: The oscillatory cosine function

left:u goes from 100 to 100.0002
right: u goes from 100 to 100.004

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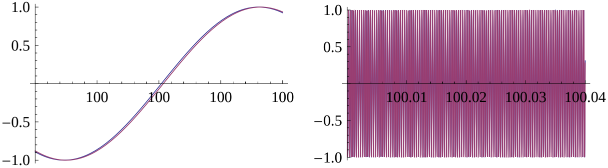

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$\overline{T}(z)$ is highly oscillatory.

- Takes hours, if not days to calculate.
  - Only for $\alpha = 1$.
  - We need to check for higher values of α.
- Similar $\overline{T}(z)$ for higher α values are bound to give more highly oscillatory integrals
  - We need systematic way to calculate these oscillatory integrals.
  - Check whether our model for potential is plausible.

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Newton-Cotes Rule

- Trapezoidal rule
- Simpson's rule

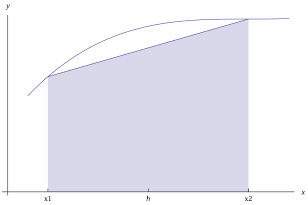

Figure: Plot showing integration by trapezoidal rule

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b) = \frac{(b-a)}{2}(f(a) + f(b)).$$

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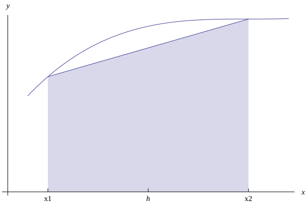

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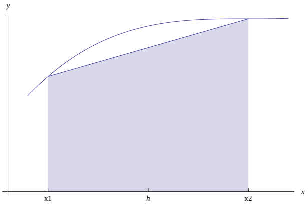

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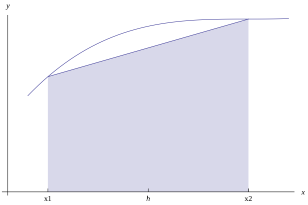

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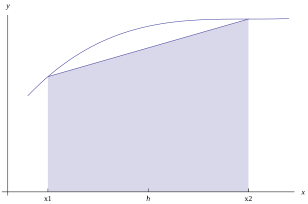

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Gauss-Quadrature

- $\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$.
- Here, $c_1, c_2, x_1,$ and x_2 are all unknowns.
- In this case, these four constants are found by integrating third order polynomials and equating the coefficients.

$$x_1 = \frac{b-a}{2}\frac{-1}{\sqrt{3}} + \frac{b+a}{2},$$

$$x_2 = \frac{b-a}{2}\frac{1}{\sqrt{3}} + \frac{b+a}{2},$$

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Filon's method

$$\int_a^b f(x)\sin\omega x dx \text{ and } \int_0^\infty \frac{f(x)}{x}\sin\omega x$$

$$\int f(x)\sin(\omega x)dx = \sum_{m=\mu}^{2\mu+2} f(x)\sin(\omega x)$$

$$m_\mu(x_\mu) = f(x_\mu), m_{\mu+1}(x_{\mu+1}) = f(x_{\mu+1}), \text{ and } m_{\mu+2}(x_{\mu+2}) = f(x_{\mu+2}).$$

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ClenshawCurtis Method

C.W.Clenshaw and A.R. Curtis in 1960. Expand f(x) in Chebyshev polynomials.

$$f(x) = F(t) = \frac{1}{2}a_0 + a_1 T_1(t) + a_2 T_2(t) + ... + \frac{1}{2}a_n T_n(t), (a \leq x \leq b) \tag{2}$$

where,

$$T_n(t) = \cos(n \cos^{-1}(t), \; t = \frac{2x - (b+a)}{b-a} \tag{3}$$

and this eventually reduces to

$$f(x) = \frac{a_0}{2} T_0(x) + \sum_{n=1}^{\infty} a_n T_n(x), x_n = cos(\frac{n\pi}{N}). \tag{4}$$

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Levin-Iserles′ Method

Improvement over Filon's method

$$
\begin{aligned}
Q_2^F[f] = & \left(-\frac{1}{i\omega} - 6\frac{1+e^{i\omega}}{i\omega^3} + 12\frac{1-e^{i\omega}}{\omega^4}\right) f(0) \\
& + \left(\frac{e^{i\omega}}{i\omega} + 6\frac{1+e^{i\omega}}{i\omega^3} - 12\frac{1-e^{i\omega}}{\omega^4}\right) f(1) \\
& + \left(-\frac{1}{\omega^2} - 2\frac{2+e^{i\omega}}{i\omega^3} + 6\frac{1-e^{i\omega}}{\omega^4}\right) f'(0) \\
& + \left(\frac{e^{i\omega}}{\omega^2} - 2\frac{1+e^{i\omega}}{i\omega^3} + 6\frac{1-e^{i\omega}}{\omega^4}\right) f'(1)
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Was it worth the time?

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- Iserles' method did not work for our $\overline{T}(z)$ integral.
- Were able to calculate integrals much faster.
- Not very consistent.

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What quadrature method to choose?

- Levin-Iserles′ method seems more promising.
- Clenshawcurtis' qudrature method also works well.

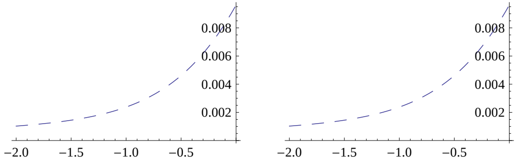

Figure: $\overline{T}(z)$using Levin and Clenshawcurtis method

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- choose methods judiciously.
- Reduce error for integrands with large frequency.

- Outlook
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NSF−0554849 and PHY−0968269.

For Further Reading I

- J. D. Bouas and S. A. Fulling and F. D. Mera and K. Thapa and C. S. Trendafilova and and J. Wagner
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