Calculation of Highly Oscillatory Integrals by Quadrature Methods

Krishna Thapa
Department of Physics & Astronomy
Texas A&M University
College Station, TX

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1 Motivation
- Study of Vacuum Energy
- Oscillatory Integrals
- Earlier Literature

2 Our Results
- Main Results
- Implementation
Motivation

Study of Vacuum Energy

Oscillatory Integrals

Earlier Literature

Outline

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A model for Vacuum Energy

Our model of quantum vacuum energy density near the boundary has the form $\lambda z^\alpha$.

**Figure:** Steeply rising potential near the boundary
Spectral analysis of the rising potential gives Energy momentum tensor:

\[
\bar{T}(z) = \frac{1}{\pi^3} \int_0^\infty d\rho \int_0^1 du \sqrt{1 - u^2} \cos(2z\rho u - 2\delta(\rho u))
\]

where,

\[
\delta(u) = \text{ArcTan}\left(-u \left(\frac{\text{AiryAi}(-u^2)}{\text{AiryAi}'(-u^2)}\right)\right)
\]
What does it look like?

**Figure:** The oscillatory cosine function

left: $u$ goes from 100 to 100.0002
right: $u$ goes from 100 to 100.004
Figure: The oscillatory cosine function

left: $u$ goes from 100 to 100.0002
right: $u$ goes from 100 to 100.004
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\( \bar{T}(z) \) is highly oscillatory.

- Takes hours, if not days to calculate.
  - Only for \( \alpha = 1 \).
  - We need to check for higher values of \( \alpha \).
- Similar \( \bar{T}(z) \) for higher \( \alpha \) values are bound to give more highly oscillatory integrals
  - We need systematic way to calculate these oscillatory integrals.
  - Check whether our model for potential is plausible.
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Motivation

Our Results

Summary

Study of Vacuum Energy

Oscillatory Integrals

Earlier Literature

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$\overline{T}(z)$ is highly oscillatory.

- Takes hours, if not days to calculate.
  - Only for $\alpha = 1$.
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- Similar $\overline{T}(z)$ for higher $\alpha$ values are bound to give more highly oscillatory integrals
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Newton-Cotes Rule

- Trapezoidal rule
- Simpson’s rule

\[ \int_{a}^{b} f(x) \, dx \approx c_1 f(a) + c_2 f(b) = \frac{b - a}{2} (f(a) + f(b)). \]
Newton-Cotes Rule

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Figure: Plot showing integration by trapezoidal rule

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Gauss-Quadrature

\[ \int_a^b f(x) \, dx \approx c_1 f(x_1) + c_2 f(x_2). \]
Here, \( c_1, c_2, x_1, \) and \( x_2 \) are all unknowns.
In this case, these four constants are found by integrating third order polynomials and equating the coefficients.

\[
x_1 = \frac{b - a - 1}{2} \frac{1}{\sqrt{3}} + \frac{b + a}{2},
\]

\[
x_2 = \frac{b - a}{2} \frac{1}{\sqrt{3}} + \frac{b + a}{2},
\]

\[ c_1 = \frac{b - a}{2}, \text{ and } c_2 = \frac{b - a}{2}. \]
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\begin{align*}
  x_1 &= \frac{b - a - 1}{2} \sqrt{3} + \frac{b + a}{2}, \\
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Gauss-Quadrature

- \( \int_{a}^{b} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) \).
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\[
x_1 = \frac{b - a}{2} - \frac{1}{\sqrt{3}} + \frac{b + a}{2},
\]
\[
x_2 = \frac{b - a}{2} + \frac{1}{\sqrt{3}} + \frac{b + a}{2},
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c_1 = \frac{b - a}{2}, \text{ and } c_2 = \frac{b - a}{2}.
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Filon’s method

\[
\int_a^b f(x) \sin \omega x \, dx \quad \text{and} \quad \int_0^\infty \frac{f(x)}{x} \sin \omega x \, dx
\]

\[
\int f(x) \sin(\omega x) \, dx = \sum_{m=\mu}^{2\mu+2} f(x) \sin(\omega x)
\]

\[
m_\mu(x_\mu) = f(x_\mu), \quad m_{\mu+1}(x_{\mu+1}) = f(x_{\mu+1}), \quad \text{and} \quad m_{\mu+2}(x_{\mu+2}) = f(x_{\mu+2}).
\]

\[
\int_a^b f(x) \sin \omega x \, dx \approx \sum_{\mu=0}^{n-1} \int_{x_{2\mu}}^{x_{2\mu+2}} m_\mu(x) \sin \omega x \, dx
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\[
f(x) = F(t) = \frac{1}{2} a_0 + a_1 T_1(t) + a_2 T_2(t) + \ldots + \frac{1}{2} a_n T_n(t), (a \leq x \leq b)
\]

(2)

where,

\[
T_n(t) = \cos(n \cos^{-1}(t)), \quad t = \frac{2x - (b + a)}{b - a}
\]

(3)

and this eventually reduces to

\[
f(x) = \frac{a_0}{2} T_0(x) + \sum_{n=1}^{\infty} a_n T_n(x), \quad x_n = \cos\left(\frac{n\pi}{N}\right).
\]

(4)
Clenshaw-Curtis Method


$$f(x) = F(t) = \frac{1}{2}a_0 + a_1 T_1(t) + a_2 T_2(t) + \ldots + \frac{1}{2}a_n T_n(t), \quad (a \leq x \leq b) \quad (2)$$

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Clenshaw-Curtis Method


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Levin-Iserles’ Method

Improvement over Filon’s method

\[
Q_2^F[f] = \left( -\frac{1}{i\omega} - 6\frac{1 + e^{i\omega}}{i\omega^3} + 12\frac{1 - e^{i\omega}}{\omega^4} \right) f(0) \\
+ \left( e^{i\omega} - 6\frac{1 + e^{i\omega}}{i\omega^3} - 12\frac{1 - e^{i\omega}}{\omega^4} \right) f(1) \\
+ \left( -\frac{1}{\omega^2} - 2\frac{2 + e^{i\omega}}{i\omega^3} + 6\frac{1 - e^{i\omega}}{\omega^4} \right) f'(0) \\
+ \left( e^{i\omega} - 2\frac{1 + e^{i\omega}}{i\omega^3} + 6\frac{1 - e^{i\omega}}{\omega^4} \right) f'(1)
\]  

(5)
Levin-Iserles’ Method

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Q_2^F[f] = \left(-\frac{1}{i\omega} - 6\frac{1 + e^{i\omega}}{i\omega^3} + 12\frac{1 - e^{i\omega}}{\omega^4}\right)f(0) \\
+ \left(\frac{e^{i\omega}}{i\omega} + 6\frac{1 + e^{i\omega}}{i\omega^3} - 12\frac{1 - e^{i\omega}}{\omega^4}\right)f(1) \\
+ \left(-\frac{1}{\omega^2} - 2\frac{2 + e^{i\omega}}{i\omega^3} + 6\frac{1 - e^{i\omega}}{\omega^4}\right)f'(0) \\
+ \left(\frac{e^{i\omega}}{\omega^2} - 2\frac{1 + e^{i\omega}}{i\omega^3} + 6\frac{1 - e^{i\omega}}{\omega^4}\right)f'(1)
\] (5)
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Was it worth the time?

- Yes, and No.
- Iserles’ method did not work for our $\overline{T}(z)$ integral.
- Were able to calculate integrals much faster.
- Not very consistent.
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Krishna Thapa Department of Physics & Astronomy Texas A&M

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What quadrature method to choose?

- Levin-Iserles’ method seems more promising.
- Clenshaw-Curtis’ quadrature method also works well.

*Figure: $\overline{T}(z)$ using Levin and Clenshaw-Curtis method*
Summary

- Highly oscillatory integrals can be calculated much faster than by conventional methods.
- Choose methods judicially.
- Reduce error for integrands with large frequency.

Outlook
- Not enough data for conclusion.
- Check for higher values of $\alpha$. 
Highly oscillatory integrals can be calculated much faster than by conventional methods.

choose methods judiciously.

Reduce error for integrands with large frequency.

Outlook

Not enough data for conclusion.

Check for higher values of $\alpha$. 
NSF–0554849 and PHY–0968269.
For Further Reading I

J. D. Bouas and S. A. Fulling and F. D. Mera and K. Thapa and C. S. Trendafilova and J. Wagner
*Investigating the Spectral Geometry of a Soft Wall*

Arieh Iserles and Syvert P. Norsett
*Quadrature methods for multivariate highly oscillatory integrals using derivatives.*

Louis Napoleon George Filon
*On a quadrature formula for trigonometric integrals*