Boundary conditions on a δ-function material

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Motivation

In my last year's talk I presented thin-plate limit of the finite thickness d material slab to obtain a δ-function plate.
For planar geometry potential is

$$V(z) = (\varepsilon_i - 1)\left[\theta(z-a) - \theta(z-a-d)\right],$$

To get a δ-function potential for $d \to 0$ we require that

$$(\varepsilon - 1) \propto \frac{1}{d}$$

One of the goals was to "try solving for Green's dyadic for δ-function potential again".
In this talk I present the recently completed work on this.

Outline

Semi-transparent delta plates

We consider an idealized infinitesimally thin material whose electric and magnetic properties are described by

$$\boldsymbol{\varepsilon}(z) - \mathbf{1} = \boldsymbol{\lambda}_e \delta(z - a),$$
$$\boldsymbol{\mu}(z) - \mathbf{1} = \boldsymbol{\lambda}_g \delta(z - a).$$

We assume that the electric permittivity and the magnetic permeability is isotropic in the plane of the plate only: $\boldsymbol{\varepsilon} = \text{diag}(\varepsilon^{\perp}, \varepsilon^{\perp}, \varepsilon^{||})$ and $\boldsymbol{\mu} = \text{diag}(\mu^{\perp}, \mu^{\perp}, \mu^{||})$.
Due to the rotational symmetry about the normal to the plate the Maxwell's equations

$$\boldsymbol{\nabla} \times \mathbf{E} = i\omega \mathbf{B},$$
$$-\boldsymbol{\nabla} \times \mathbf{H} = i\omega (\mathbf{D} + \mathbf{P}),$$

decouples into transverse electric and transverse magnetic modes.

TM or E-mode (H_1, E_2, H_3):

$$
\begin{aligned}
H_2(z) &= -\frac{\omega}{k_\perp} D_3(z) - \frac{\omega}{k_\perp} P_3(z), \\
\frac{\partial}{\partial z} D_3(z) &= -ik_\perp D_1(z) - ik_\perp P_1(z) - \frac{\partial}{\partial z} P_3(z), \\
\frac{\partial}{\partial z} E_1(z) &= ik_\perp E_3(z) + i\omega B_2(z),
\end{aligned}
$$

TE or H-mode (H_1, E_2, H_3):

$$
\begin{aligned}
E_2(z) &= \frac{\omega}{k_\perp} B_3(z), \\
\frac{\partial}{\partial z} B_3(z) &= -ik_\perp B_1(z), \\
\frac{\partial}{\partial z} H_1(z) &= ik_\perp H_3(z) - i\omega D_2(z) - i\omega P_2(z).
\end{aligned}
$$

Boundary conditions

Boundary conditions are derived by integrating Maxwell's equations. We require

$$\lim_{\delta\to 0}\int_{a-\delta}^{a+\delta} dz\,\mathbf{E}(z) = 0, \qquad \text{and} \qquad \lim_{\delta\to 0}\int_{a-\delta}^{a+\delta} dz\,\mathbf{H}(z) = 0.$$

The boundary conditions for TM mode are:

$$\begin{aligned} \lambda_e^{||} E_3(a) &= 0, \\ D_3(a+\delta) - D_3(a-\delta) &= -ik_\perp \lambda_e^\perp E_1(a), \\ E_1(a+\delta) - E_1(a-\delta) &= i\omega\lambda_g^\perp H_2(a), \end{aligned}$$

and the boundary conditions for TE mode are:

$$\begin{aligned} \lambda_g^{||} H_3(a) &= 0, \\ B_3(a+\delta) - B_3(a-\delta) &= -ik_\perp \lambda_g^\perp H_1(a), \\ H_1(a+\delta) - H_1(a-\delta) &= -i\omega\lambda_e^\perp E_2(a). \end{aligned}$$

Combining the first order differential equations yields-

$$\left[-\frac{\partial}{\partial z}\frac{1}{\varepsilon^{\perp}(z)}\frac{\partial}{\partial z}+\frac{k_{\perp}^{2}}{\varepsilon^{||}(z)}-\omega^{2}\mu^{\perp}(z)\right]H_{2}(z)=-i\omega\frac{\partial}{\partial z}\frac{P_{1}(z)}{\varepsilon^{\perp}(z)}-\omega k_{\perp}\frac{P_{3}(z)}{\varepsilon^{||}(z)},$$

$$\left[-\frac{\partial}{\partial z}\frac{1}{\mu^{\perp}(z)}\frac{\partial}{\partial z}+\frac{k_{\perp}^{2}}{\mu^{||}(z)}-\omega^{2}\varepsilon^{\perp}(z)\right]E_{2}(z)=\omega^{2}P_{2}(z).$$

The remaining field components can be expressed in terms of $H_2(z)$ and $E_2(z)$.

Fields and Green's functions

We define the magnetic Green's function $g^H(z, z')$, and the electric Green's function $g^E(z, z')$, as the inverse of the differential operators, to construct

$$\left[-\frac{\partial}{\partial z}\frac{1}{\varepsilon^{\perp}(z)}\frac{\partial}{\partial z} + \frac{k_{\perp}^2}{\varepsilon^{||}(z)} - \omega^2\mu^{\perp}(z)\right] g^H(z, z') = \delta(z - z'),$$
$$\left[-\frac{\partial}{\partial z}\frac{1}{\mu^{\perp}(z)}\frac{\partial}{\partial z} + \frac{k_{\perp}^2}{\mu^{||}(z)} - \omega^2\varepsilon^{\perp}(z)\right] g^E(z, z') = \delta(z - z'),$$

The fields are expressed in terms of Green's dyadics

$$\mathbf{E}(z) = \int dz'\, \gamma(z,z') \cdot \mathbf{P}(z'),$$
$$\mathbf{H}(z) = \int dz'\, \phi(z,z') \cdot \mathbf{P}(z'),$$

where

$$\gamma(z,z') = \begin{bmatrix} \frac{1}{\varepsilon^{\perp}}\frac{\partial}{\partial z}\frac{1}{\varepsilon'^{\perp}}\frac{\partial}{\partial z'} g^H & 0 & \frac{1}{\varepsilon^{\perp}}\frac{\partial}{\partial z}\frac{ik_\perp}{\varepsilon'^{||}} g^H \\ 0 & \omega^2 g^E & 0 \\ -\frac{ik_\perp}{\varepsilon^{||}(z)}\frac{1}{\varepsilon'^{\perp}}\frac{\partial}{\partial z'} g^H & 0 & -\frac{ik_\perp}{\varepsilon^{||}}\frac{ik_\perp}{\varepsilon'^{||}} g^H \end{bmatrix} - \delta(z-z') \begin{bmatrix} \frac{1}{\varepsilon^{\perp}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\varepsilon^{||}} \end{bmatrix}$$

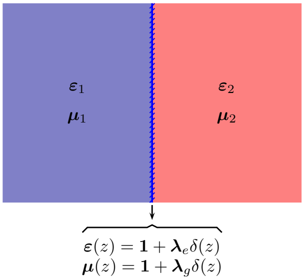
and

$$\phi(z,z') = i\omega \begin{bmatrix} 0 & \frac{1}{\mu^{\perp}}\frac{\partial}{\partial z} g^E & 0 \\ \frac{1}{\varepsilon'^{\perp}}\frac{\partial}{\partial z'} g^H & 0 & \frac{ik_\perp}{\varepsilon'^{||}} g^H \\ 0 & -\frac{ik_\perp}{\mu^{||}(z)} g^E & 0 \end{bmatrix}.$$

Magnetic and Electric Green's functions

Consider δ-function plate sandwiched between two uniaxial materials, described by

$$\boldsymbol{\varepsilon}(z) = \varepsilon^{\perp}(z)\,\mathbf{1}_{\perp} + \varepsilon^{||}(z)\,\hat{\mathbf{z}}\,\hat{\mathbf{z}},$$
$$\boldsymbol{\mu}(z) = \mu^{\perp}(z)\,\mathbf{1}_{\perp} + \mu^{||}(z)\,\hat{\mathbf{z}}\,\hat{\mathbf{z}},$$



where

$$\varepsilon^{\perp,||}(z) = 1 + (\varepsilon_1^{\perp,||} - 1)\theta(a - z) + (\varepsilon_2^{\perp,||} - 1)\theta(z - a) + \lambda_e^{\perp,||}\delta(z - a),$$
$$\mu^{\perp,||}(z) = 1 + (\mu_1^{\perp,||} - 1)\theta(a - z) + (\mu_2^{\perp,||} - 1)\theta(z - a) + \lambda_g^{\perp,||}\delta(z - a).$$

Boundary conditions on scalar Green's functions

The boundary conditions on the TM mode in terms of reduced Green's dyadic are

$$\varepsilon_2^{||}\gamma_{3i}(a+\delta,z') - \varepsilon_1^{||}\gamma_{3i}(a-\delta,z') = -ik_\perp\lambda_e^\perp\frac{1}{2}\big[\gamma_{1i}(a+\delta,z')+\gamma_{1i}(a-\delta,z')\big],$$

$$\gamma_{1i}(a+\delta,z') - \gamma_{1i}(a-\delta,z') = i\omega\lambda_g^\perp\frac{1}{2}\big[\phi_{2i}(a+\delta,z')+\phi_{2i}(a-\delta,z')\big],$$

which in terms of scalar Green's function are

$$g^H(z,z')\Big|_{z=a-\delta}^{z=a+\delta} = \frac{\lambda_e^\perp}{2}\left[\left\{\frac{1}{\varepsilon^\perp(z)}\frac{\partial}{\partial z}g^H\right\}_{z=a+\delta} + \left\{\frac{1}{\varepsilon^\perp(z)}\frac{\partial}{\partial z}g^H\right\}_{z=a-\delta}\right],$$

$$\left\{\frac{1}{\varepsilon^\perp(z)}\frac{\partial}{\partial z}g^H(z,z')\right\}\Bigg|_{z=a-\delta}^{z=a+\delta} = \zeta^2\frac{\lambda_g^\perp}{2}\big[g^H(a+\delta,z')+g^H(a-\delta,z')\big].$$

Corresponding boundary conditions on the TE mode gives

$$\mu_2^{||}\,\phi_{3i}(a+\delta,z') - \mu_1^{||}\,\phi_{3i}(a-\delta,z') = -ik_\perp\lambda_g^\perp\frac{1}{2}\big[\phi_{1i}(a+\delta,z')+\phi_{1i}(a-\delta,z')\big],$$

$$\phi_{1i}(a+\delta,z') - \phi_{1i}(a-\delta,z') = -i\omega\lambda_e^\perp\frac{1}{2}\big[\gamma_{2i}(a+\delta,z')+\gamma_{2i}(a-\delta,z')\big].$$

In terms of scalar green's functions these are

$$g^E(z,z')\Big|_{z=a-\delta}^{z=a+\delta} = \frac{\lambda_g^\perp}{2}\left[\left\{\frac{1}{\mu^\perp(z)}\frac{\partial}{\partial z}g^E\right\}_{z=a+\delta} + \left\{\frac{1}{\mu^\perp(z)}\frac{\partial}{\partial z}g^E\right\}_{z=a-\delta}\right],$$

$$\left\{\frac{1}{\mu^\perp(z)}\frac{\partial}{\partial z}g^E(z,z')\right\}\Bigg|_{z=a-\delta}^{z=a+\delta} = \zeta^2\frac{\lambda_e^\perp}{2}\big[g^E(a+\delta,z')+g^E(a-\delta,z')\big].$$

Solution for the scalar Green's function

The solution for the magnetic Green's function satisfying the boundary conditions is

$$g^H(z,z') = \begin{cases} \frac{1}{2\bar{\kappa}_1^H}\left[e^{-\kappa_1^H|z-z'|} + r_{12}^H\, e^{-\kappa_1^H|z-a|}e^{-\kappa_1^H|z'-a|}\right], & \text{if} \quad z,z'<a, \\ \frac{1}{2\bar{\kappa}_2^H}\left[e^{-\kappa_2^H|z-z'|} + r_{21}^H\, e^{-\kappa_2^H|z-a|}e^{-\kappa_2^H|z'-a|}\right], & \text{if} \quad a<z,z', \\ \frac{1}{2\bar{\kappa}_2^H}\, t_{21}^H\, e^{-\kappa_1^H|z-a|}e^{-\kappa_2^H|z'-a|}, & \text{if} \quad z<a<z', \\ \frac{1}{2\bar{\kappa}_1^H}\, t_{12}^H\, e^{-\kappa_2^H|z-a|}e^{-\kappa_1^H|z'-a|}, & \text{if} \quad z'<a<z, \end{cases}$$

where

$$\kappa_i^H = \sqrt{k_\perp^2 \frac{\varepsilon_i^\perp}{\varepsilon_i^{||}} + \zeta^2 \varepsilon_i^\perp \mu_i^\perp} \qquad \text{and} \qquad \bar{\kappa}_i^H = \frac{\kappa_i^H}{\varepsilon_i^\perp} = \sqrt{\frac{k_\perp^2}{\varepsilon_i^\perp \varepsilon_i^{||}} + \zeta^2 \frac{\mu_i^\perp}{\varepsilon_i^\perp}}.$$

The reflection coefficients are

$$r_{ij}^H = \frac{\bar\kappa_i^H\left(1+\frac{\lambda_e^\perp\bar\kappa_j^H}{2}\right)\left(1-\frac{\lambda_g^\perp\zeta^2}{2\bar\kappa_i^H}\right)-\bar\kappa_j^H\left(1-\frac{\lambda_e^\perp\bar\kappa_i^H}{2}\right)\left(1+\frac{\lambda_g^\perp\zeta^2}{2\bar\kappa_j^H}\right)}{\bar\kappa_i^H\left(1+\frac{\lambda_e^\perp\bar\kappa_j^H}{2}\right)\left(1+\frac{\lambda_g^\perp\zeta^2}{2\bar\kappa_i^H}\right)+\bar\kappa_j^H\left(1+\frac{\lambda_e^\perp\bar\kappa_i^H}{2}\right)\left(1+\frac{\lambda_g^\perp\zeta^2}{2\bar\kappa_j^H}\right)},$$

and the transmission coefficients are

$$t_{ij}^H = \frac{\bar\kappa_i^H\left(1+\frac{\lambda_e^\perp\bar\kappa_i^H}{2}\right)\left(1-\frac{\lambda_g^\perp\zeta^2}{2\bar\kappa_i^H}\right)+\bar\kappa_i^H\left(1-\frac{\lambda_e^\perp\bar\kappa_i^H}{2}\right)\left(1+\frac{\lambda_g^\perp\zeta^2}{2\bar\kappa_i^H}\right)}{\bar\kappa_i^H\left(1+\frac{\lambda_e^\perp\bar\kappa_j^H}{2}\right)\left(1+\frac{\lambda_g^\perp\zeta^2}{2\bar\kappa_i^H}\right)+\bar\kappa_j^H\left(1+\frac{\lambda_e^\perp\bar\kappa_i^H}{2}\right)\left(1+\frac{\lambda_g^\perp\zeta^2}{2\bar\kappa_j^H}\right)}.$$

The electric Green's function is obtained from the magnetic Green's function by replacing $\boldsymbol{\varepsilon} \leftrightarrow \boldsymbol{\mu}$ and $H \to E$, with

$$\kappa_i^E = \sqrt{k_\perp^2\frac{\mu_i^\perp}{\mu_i^{||}} + \zeta^2\mu_i^\perp\varepsilon_i^\perp} \qquad \text{and} \qquad \bar\kappa_i^E = \frac{\kappa_i^E}{\mu_i^\perp} = \sqrt{\frac{k_\perp^2}{\mu_i^\perp\mu_i^{||}} + \zeta^2\frac{\varepsilon_i^\perp}{\mu_i^\perp}}.$$

Green's functions for a semitransparent δ-function plate

A semitransparent δ-function plate in vacuum corresponds to setting $\varepsilon_i^\perp = \varepsilon_i^{||} = 1$ and $\mu_i^\perp = \mu_i^{||} = 1$.
The magnetic Green's function is

$$g^H(z,z') = \frac{1}{2\kappa} e^{-\kappa|z-z'|} + \left[r_g^H + \eta(z-a)\eta(z'-a)\, r_e^H \right] \frac{1}{2\kappa} e^{-\kappa|z-a|} e^{-\kappa|z'-a|}.$$

These reflection coefficients, r_e^H and r_g^H, and the corresponding transmission coefficients, t_e^H and t_g^H, are related by,

$$r_e^H = \frac{\lambda_e^\perp}{\lambda_e^\perp + \frac{2}{\kappa}}, \quad t_e^H = 1 - r_e^H, \qquad \text{and} \qquad r_g^H = -\frac{\lambda_g^\perp}{\lambda_g^\perp + \frac{2\kappa}{\zeta^2}}, \quad t_g^H = 1 + r_g^H.$$

The total reflection and transmission coefficients for the magnetic mode, with reference to Eqs. (16)), are

$$r^H = r_g^H + r_e^H, \qquad t^H = 1 + r_g^H - r_e^H.$$

Conditions implying anisotropy

Consider the first boundary conditions $\lambda_e^{||} E_3(a) = 0$ and $\lambda_g^{||} H_3(a) = 0$. Left hand side evaluates to

$$\lambda_e^{||} E_3(a) = -\frac{ik_\perp}{2}(1 + r_g^H)\lambda_e^{||} = -\frac{ik_\perp}{2}\frac{2\kappa}{\zeta^2}\frac{\lambda_e^{||}}{\left(\lambda_g^\perp + \frac{2\kappa}{\zeta^2}\right)} = 0,$$

$$\lambda_g^{||} H_3(a) = -\frac{ik_\perp}{2}(1 + r_e^H)\lambda_g^{||} = -\frac{ik_\perp}{2}\frac{2\kappa}{\zeta^2}\frac{\lambda_g^{||}}{\left(\lambda_e^\perp + \frac{2\kappa}{\zeta^2}\right)} = 0.$$

The sufficient condition for a δ-function plate to be anisotropic is

$$\frac{\lambda_e^{||}}{\lambda_g^\perp} = 0 \qquad \text{and} \qquad \frac{\lambda_g^{||}}{\lambda_e^\perp} = 0.$$

Thin plate limit

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In what approximation will a dielectric slab of thickness d simulate a (purely electric) semitransparent δ-function plate?

$$\varepsilon^{\perp}(i\zeta) - 1 = \lambda_e^{\perp}(i\zeta) \lim_{d\to 0} \frac{[\theta(z+d) - \theta(z)]}{d},$$

which in the limit $d \to 0$ gives the δ-function response of dielectric permittivity. It describes a dielectric slab of thickness d if we read the factor $\lambda_e^{\perp}(i\zeta)/d$ to represent the slab's susceptibility.
In plasma model we can consider frequency response of $\lambda_e^{\perp}(i\zeta)$ as

$$\lambda_e^{\perp}(i\zeta) = \frac{\zeta_p}{\zeta^2},$$

Now if we impose the following conditions for thin-plate limit

$$\zeta^2 \ll \frac{\zeta_p}{d} \ll \frac{1}{d^2}, \qquad \text{and} \qquad k_\perp^2 \ll \frac{\zeta_p}{d} \ll \frac{1}{d^2},$$

The reflection coefficients for the TM–and TE-modes for a dielectric slab of thickness d has the following limiting behavior,

$$r^H_{\text{thick}} = -\left(\frac{\bar{\kappa}^H_i - \kappa}{\bar{\kappa}^H_i + \kappa}\right) \frac{(1 - e^{-2\kappa^H_i d})}{\left[1 - \left(\frac{\bar{\kappa}^H_i - \kappa}{\bar{\kappa}^H_i + \kappa}\right)^2 e^{-2\kappa^H_i d}\right]} \xrightarrow[k_\perp d \ll \sqrt{\zeta_p d} \ll 1]{\zeta d \ll \sqrt{\zeta_p d} \ll 1} r^H_e = \frac{\lambda^\perp_e(i\zeta)}{\lambda^\perp_e(i\zeta) + \frac{2}{\kappa}},$$

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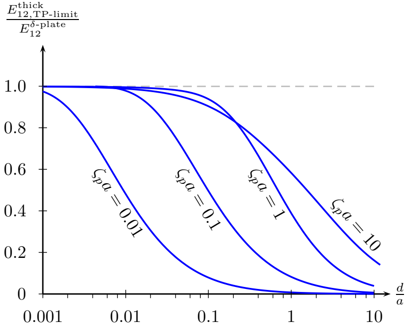
Rearranging the limiting conditions, shows that thin-plate limit is a good approximation of the interaction energy between two δ-plates in the parameter regime

$$\frac{d}{a} \ll \zeta_p a \ll \frac{1}{d/a}.$$



Atom in front of a δ-function plate

Consider an atom described by potential $\mathbf{V}(\mathbf{x}) = 4\pi\,\boldsymbol{\alpha}(i\zeta)\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)$.

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The Casimir-Polder energy between an atom and a δ-function plate is

$$E_{12}^{\text{CP}} = -2\pi \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \int \frac{d^2k_\perp}{(2\pi)^2}\, \frac{e^{-2\kappa a}}{2\kappa} \alpha \Big[\kappa^2 r^H - \zeta^2 r^E + k_\perp^2\, r^H\Big].$$

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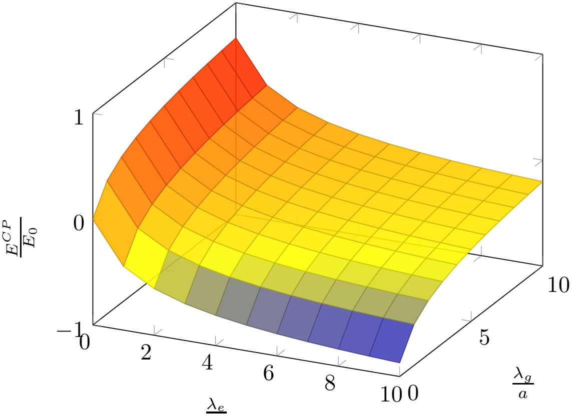

Summary and future work

- δ-function plates are necessarily anisotropic.
- The thin-plate limit of the Casimir and Casimir-Polder energies reduces to the results of the δ-function plates.
- The perfect conductor limits gives the usual Casimir and Casimir-Polder results.
- We still do not difference in see the "thin" boundary condition described by Bordag. The "thick" and "thin" propagators described in his paper corresponds to same Green's dyadic and therefore corresponds to same physical situation.
- Inclusion of magnetic properties will lead to interesting results.
- We also employed the low frequency limit of the Drude model to describe thin-plate limit to model graphene and shall continue to work on it further.

Thank you all for listening.