Topological and Quantum Aspects of Hertz Potentials

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Quantum Vacuum Workshop

Outline

- Review
- Topology
- Quantization
- Topology and Quantization

Review

Maxwell's Equations

Vector Notation

$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{B} + \partial_t \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{E} = \rho$$
$$\vec{\nabla} \times \vec{E} - \partial_t \vec{B} = \vec{J}$$

Differential Form Notation

$$dF = 0$$
$$\delta F = J$$

$$F = \frac{1}{2} F_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}$$
$$= E_i\, dt \wedge dx^{i} + B_i * (dt \wedge dx^{i})$$

Review

Charge Conservation

$$\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0 \qquad \delta J = 0$$

Electromagnetic Potentials

$$\vec{B} = \vec{\nabla} \times \vec{A}$$
$$\vec{E} = -\vec{\nabla} V - \partial_t \vec{A}$$
$$F = dA$$

Lorenz Gauge Condition

$$\partial_t V + \vec{\nabla} \cdot \vec{A} = 0 \qquad \delta A = 0$$

Review

Hertz Potentials

$$V = -\vec{\nabla} \cdot \vec{\Pi}_e$$
$$\vec{A} = \partial_t \vec{\Pi}_e + \vec{\nabla} \cdot \vec{\Pi}_m$$

$$A = \delta \Pi$$

$$\vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{\Pi}_e) - \partial_t^2 \vec{\Pi}_e - \partial_t \vec{\nabla} \times \vec{\Pi}_m$$
$$\vec{B} = \vec{\nabla} \times \partial_t \vec{\Pi}_e + \vec{\nabla} \times (\vec{\nabla} \cdot \vec{\Pi}_m)$$

$$F = d\delta \Pi$$

Review

Special Geometries

Cylindrical Symmetry

$$\vec{\Pi}_e = \phi\,\hat{z}$$
$$\vec{\Pi}_m = \psi\,\hat{z}$$

$$\Pi = \phi\,dt{\wedge}dz + \psi\,*(dt{\wedge}dz)$$

Spherical Symmetry

$$\vec{\Pi}_e = \phi\,\vec{r}$$
$$\vec{\Pi}_m = \psi\,\vec{r}$$

$$\Pi = \phi \cdot r\,dt{\wedge}dr + \psi \cdot r\,*(dt{\wedge}dr)$$

Review

Equations of Motion

$$\Box\Pi = dG + \delta W$$

Cylindrical Symmetry

$$G = 0,\ W = 0$$

$$\Box\Pi = 0$$

$$\Box\phi = 0,\ \Box\psi = 0$$

Spherical Symmetry

$$G = 2\phi\, dt,\ *W = 2\psi\, dt$$

$$\Box\Pi = dG + \delta W$$

$$\Box\phi = 0,\ \Box\psi = 0$$

Review

Boundary Conditions

Arbitrary Domain

$$\hat{n} \cdot \vec{B} = 0 \qquad \vec{E}_{\parallel} = 0$$

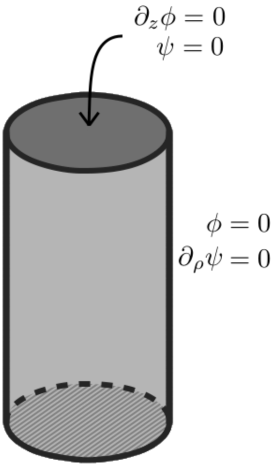
Cylindrically Symmetric Domain

On the caps

$$\partial_z \phi = 0 \qquad \psi = 0$$

On the cylinder

$$\phi = 0 \qquad \partial_\rho \psi = 0$$

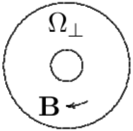


Topology

Coaxial Cable

There are non-vanishing modes

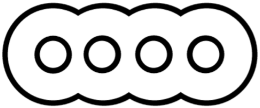
$$\vec{A} = \sin\left(\frac{\pi m z}{L}\right) e^{\pm \pi m t/L} \frac{\hat{\rho}}{\rho}$$



where no Π exists such that $A = \delta\Pi$.

$\vec{A}_\perp = \frac{\hat{\rho}}{\rho}$ is a harmonic vector field in the cross-sectional domain (annulus) but not the disk or the plane ($\vec{A}_\perp|_{\rho=0}$ is undefined).

Topology



For a cross-sectional domain with N holes there exist N similar harmonic vector fields $\vec{A}_\perp$, so

$$\vec{A}_m = \vec{A}_\perp(\vec{x}_\perp) \sin\left(\frac{\pi m z}{L}\right) e^{\pm \pi m t/L}$$

generate a family of fields for which no Hertz potential representation exists.
All other modes can be described by Hertz potentials.

Topology

Concentric Spherical Shells

For perfectly conducting spherical shells, the field solution

$$\vec{E} = \frac{\hat{r}}{r^2},\ \vec{B} = 0$$

satisfies the boundary conditions. This of course corresponds to a net charge enclosed by the inner sphere and is but one solution rather than a family.

Topology

However, consider now "infinitely permeable" ($\mu = \infty$) shells with boundary conditions dual to perfect conductor conditions:

$$\hat{n} \cdot \vec{E} = 0 \qquad\qquad \vec{B}_{\parallel} = 0$$

This is solved by a similar field to above but with the roles of $\vec{E}$ and $\vec{B}$ reversed:

$$\vec{B} = \frac{\hat{r}}{r^2},\ \vec{E} = 0,$$

and it is this potential for which no vector potential $\vec{A}$ exists. Analogously to the "generalized" coaxial cable, such a field solution defying description in terms of potentials exists for each hole cut out of the domain.

Quantization

For now return to trivial topology.

Let $\{\phi_j, \omega_j^2\}$ be the normalized eigenfunctions and eigenvalues of $-\nabla^2$ in the cavity. Our quantization takes the form

$$\phi(t, \vec{r}) = \sum_j \frac{1}{\sqrt{2\omega_j}} [a_j \phi_j(\vec{r}) e^{-i\omega_j t} + a_j^\dagger \phi_j(\vec{r}) e^{i\omega_j t}]$$

where $[a_j, a_k^\dagger] = \delta_{jk}$, $[a_j, a_k] = 0$.

Quantization

For a complete field, we must check that

$$[E_i(t,\vec{r}), B_j(t,\vec{r}\,')] = -i\epsilon_{ij}{}^k \frac{\partial}{\partial x^k}\delta(\vec{r}-\vec{r}\,')$$

is satisfied by our constructions of ϕ and ψ.

Quantization

Topologically Trivial Cylinder

Recall $\Box\phi = 0 = \Box\psi$ with alternating boundary conditions on end caps and the cylinder. Then ϕ and ψ decompose into products of axial and cross-sectional functions where

$$\phi_{jm} = f_j(\vec{r}_\perp)\cos\left(\frac{\pi m z}{L}\right)e^{-it\omega_{jmTM}}$$

$$\psi_{jm} = g_j(\vec{r}_\perp)\sin\left(\frac{\pi m z}{L}\right)e^{-it\omega_{jmTE}}$$

where f_j and g_j solve

$$\nabla_\perp^2 f_j = -\lambda_{jD}^2 f_j,\ f_j = 0 \text{ on the boundary}$$

$$\nabla_\perp^2 g_j = -\lambda_{jN}^2 g_j,\ \partial_\rho g_j = 0 \text{ on the boundary}$$

and

$$\omega_{jmTM}{}^2 = \lambda_{jD}{}^2 + \left(\tfrac{\pi m}{L}\right)^2$$
$$\omega_{jmTE}{}^2 = \lambda_{jN}{}^2 + \left(\tfrac{\pi m}{L}\right)^2$$

Quantization

Topologically Trivial Cylinder

1. Vanishing commutators (e.g. $[E_z, B_z]$, $[E_z, E_z]$, $[E_z, \vec{E}_\perp]$, and $[\vec{E}_\perp, \vec{E}_\perp]$) are relatively easy to prove.

2. Axial commutators such as

$$[E_z(t, \vec{r}),\ \vec{B}_\perp(t, \vec{r}\,')] = -i \star \vec{\nabla}_\perp \delta(\vec{r} - \vec{r}\,')$$

are slightly more complicated and follow from completeness relations between the $\{f\}$, $\{\cos\}$ and $\{g\}$, $\{\sin\}$ bases in their function space.

Quantization

3. Commutators involving purely cross-sectional terms (i.e. $[\vec{E}_\perp, \vec{B}_\perp]$) require particular relationships between the Green functions in the disk, namely

$$\partial_\varphi \partial_{\rho'} G_N(\vec{r}_\perp, \vec{r}'_\perp) = \partial_\rho \partial_{\varphi'} G_D(\vec{r}_\perp, \vec{r}'_\perp),$$
$$\partial_\rho \partial_{\varphi'} G_N(\vec{r}_\perp, \vec{r}'_\perp) = \partial_\varphi \partial_{\rho'} G_D(\vec{r}_\perp, \vec{r}'_\perp),$$

and

$$\partial_\varphi \partial_{\varphi'} G_N(\vec{r}_\perp, \vec{r}'_\perp) + \partial_\rho \partial_{\rho'} G_D(\vec{r}_\perp, \vec{r}'_\perp) = \delta(\vec{r}_\perp - \vec{r}'_\perp),$$
$$\partial_\rho \partial_{\rho'} G_N(\vec{r}_\perp, \vec{r}'_\perp) + \partial_\varphi \partial_{\varphi'} G_D(\vec{r}_\perp, \vec{r}'_\perp) = \delta(\vec{r}_\perp - \vec{r}'_\perp),$$

where G_D and G_N are the Dirichlet and Neumann Green* functions.

*G_N is the Green function of the space with the subspace corresponding to the $\lambda_{0N} = 0$ eigenvalue removed.

Topology and Quantization

Coaxial Cable

Commutators from items (1) and (2) follow relatively unchanged (Bessel functions become sums of Bessel functions), however item (3) seems to break down entirely.

Recall we had a family of modes

$$\vec{A}_m = \sin\left(\frac{\pi m z}{L}\right) e^{\pm \pi m t/L} \frac{\hat{\rho}}{\rho}$$

for which no Hertz potential representation exists. Adding in the quantization from these modes yields the missing terms to complete the commutator.

Concentric Spherical Shells

Note, however, that there was no family of modes from the concentric spheres; there was just one mode. Additionally, this came purely from the electric charge enclosed by the inner shell. When quantized, such a mode is discarded as it is completely determined by the charge and cannot vary as a function of time, thus it is not a degree of freedom.

The difference between this and the coaxial cable is that whereas the sphere's charge is fixed and cannot be altered, current in the cable may fluctuate from time to time while still satisfying all charge conservation necessities, enabling the degree(s) of freedom associated with TEM modes.