

# Developments in Casimir-Polder repulsion: Three-body effects

K. A. Milton

Elom Abalo, P. Parashar, Nima Pourtolami, (OU)

Iver Brevik and Simen Ellingsen (NTNU)

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H. L. Dodge Department of Physics and Astronomy, OU

University of Oklahoma



# Introduction

We continue our analytic investigation of regimes in which repulsive Casimir and Casimir-Polder forces can occur. Previously we had considered the interaction between a polarizable atom and a wedge, and showed that repulsion occurs if the atom is sufficiently anisotropic and close enough to the symmetry plane of the wedge.

# Introduction (contd.)

Now we we are considering the interaction between such an atom and two facing wedges, which includes as a special case the the interaction of an atom with a conducting screen possessing a slit. **Three body effects are shown to be small.**

We are also considering the Casimir-Polder interaction between an atom and a conducting screen containing a circular aperture. We are examining the interaction of a atom with a conducting ellipsoid. Finally, we are considering whether such highly anisotropic atoms needed for repulsion are practically realizable.

# CP repulsion between atoms

The interaction between two polarizable atoms, described by general polarizabilities  $\alpha_{1,2}$ , with the relative separation vector given by  $r$  is

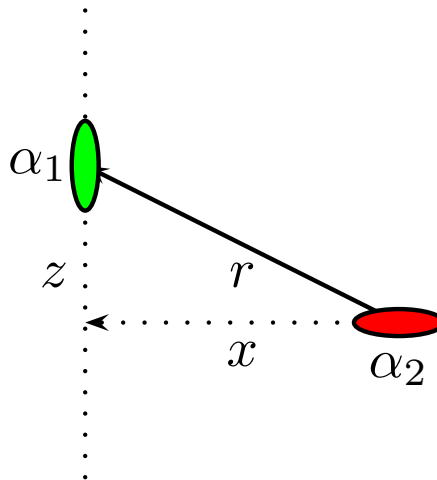
$$U_{\text{CP}} = -\frac{1}{4\pi r^7} \left[ \frac{13}{2} \text{Tr} \alpha_1 \cdot \alpha_2 - 28 \text{Tr}(\alpha_1 \cdot \hat{r})(\alpha_2 \cdot \hat{r}) + \frac{63}{2} (\hat{r} \cdot \alpha_1 \cdot \hat{r})(\hat{r} \cdot \alpha_2 \cdot \hat{r}) \right].$$

This formula is easily rederived by the multiple scattering technique. In the isotropic case,  $\alpha_i =$

$$\alpha_i \mathbf{1}, U_{\text{CP}} \rightarrow -\frac{23}{4\pi r^7} \alpha_1 \alpha_2.$$

# Anisotropic atoms

Suppose the two atoms are only polarizable in perpendicular directions,  $\alpha_1 = \alpha_1 \hat{z}\hat{z}$ ,  $\alpha_2 = \alpha_2 \hat{x}\hat{x}$ , as shown in the figure. Choose atom 2 to be at the origin.



# Force on atom

Then, in terms of the polar angle  $\cos \theta = z/r$ , the  $z$ -component of the force on atom 1 is

$$F_z = -\frac{63}{8\pi} \frac{\alpha_1 \alpha_2}{x^8} \sin^{10} \theta \cos \theta (9 - 11 \sin^2 \theta).$$

Here, for motion for fixed  $x = r \sin \theta$ , in the  $y = 0$  plane. Evidently, the force is attractive at large distances, vanishing as  $\theta \rightarrow 0$ , but it must change sign at small values of  $z$  for fixed  $x$ , since the energy also vanishes as  $\theta \rightarrow \pi/2$ .  $F_z = 0$  when  $\sin \theta = 3/\sqrt{11}$  or  $25^\circ$  from the  $x$  axis.

# Anisotropy required

No repulsion occurs if one of the atoms is isotropically polarizable. If both have cylindrically symmetric anisotropies, but with respect to perpendicular axes,

$$\alpha_1 = (1-\gamma_1)\alpha_1\hat{z}\hat{z} + \gamma_1\alpha_1\mathbf{1}, \quad \alpha_2 = (1-\gamma_2)\alpha_2\hat{x}\hat{x} + \gamma_2\alpha_2\mathbf{1},$$

it is easy to check that, if both are sufficiently anisotropic, repulsion will occur. For example, if  $\gamma_1 = \gamma_2$  repulsion in the  $z$  direction will take place close to the plane  $z = 0$  if  $\gamma \leq 0.26$ .

# Shajesh and Schaden

Shajesh and Schaden <sup>a</sup> rederived these results, and then went on to extend the calculation to Casimir-Polder repulsion by an anisotropic dilute dielectric sheet with a circular aperture. The authors quite correctly point out that the statement in our first Repulsion paper, that no repulsion is possible in the weak-coupling regime, is erroneous; all that is required is anisotropy.

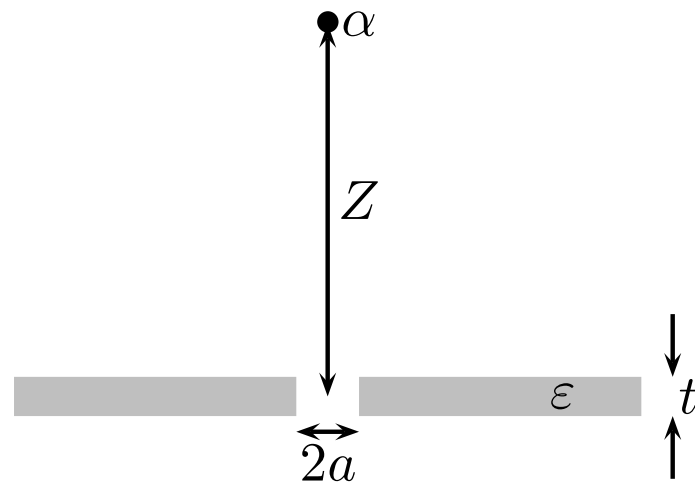
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<sup>a</sup>Shajesh and Schaden, Phys. Rev. A 85, 012523 (2012).



# Atom above aperture

We consider an anisotropic polarizable atom directly above a tenuous anisotropic slab containing a circular aperture, as shown in the figure.



# Anisotropic atom

Here we assume that the atom is only polarizable in the  $z$  direction,

$$\alpha_1 = \alpha_1 \hat{z}\hat{z},$$

while the slab is composed of atoms only transversely polarizable,

$$\alpha_2 = \alpha_2 (\hat{x}\hat{x} + \hat{y}\hat{y}).$$

For atom 1 at  $(0, 0, Z)$ , and atom 2 at  $(\rho, 0, z)$

$$U_{\text{CP}} = -\frac{63\alpha_1\alpha_2}{8\pi r^7} \left(\frac{Z-z}{r}\right)^2 \left(\frac{\rho}{r}\right)^2,$$

# Integration over slab

which when integrated over the slab made up of the type-2 atoms gives the quantum interaction energy

$$\begin{aligned} E &= -\frac{63\alpha_1\alpha_2n_2}{8\pi} \int_a^\infty d\rho\rho \int_0^{2\pi} d\theta \int_{-t/2}^{t/2} dz \frac{(Z-z)^2\rho^2}{[(Z-z)^2 + \rho^2]^{11/2}} \\ &= -\frac{63\alpha_1\alpha_2n_2}{2a^4} [e(Z/a + t/2a) - e(Z/a - t/2a)], \end{aligned}$$

where  $n_2$  is the number density of atoms, and

$$e(x) = \frac{x^3}{5} \frac{15 + 14x^2 + 4x^4}{(1 + x^2)^{7/2}}.$$

# Repulsion

Such an atom sufficiently close to the aperture experiences a repulsive force. Define a dimensionless parameter  $\delta$  that measures the height of the atom above the top of the aperture,

$$Z = \frac{t}{2} + a\delta.$$

For a thick slab,  $t/a \gg 1$ , it is easy to check that that the force changes sign very close to the opening of the aperture,

$$\delta = \frac{\sqrt{2}}{3} \left( \frac{t}{a} \right)^{-5/2} ;$$

# Thick and thin slabs

for example, when  $t/a = 10$ ,  $\delta = 1.5 \times 10^{-3}$ . When the slab is very thin,  $t/a \ll 1$ , the value of  $\delta$  for which repulsion sets in becomes independent of  $t/a$ ,  $\delta = 0.5566$ , which agrees with the result of Shajesh and Schaden.

# 3-body CP interaction

In the following we consider a polarizable atom between parallel conducting planes. We first develop the three-body interaction formalism, following Schaden and Shajesh, and then use it to show that the CP 3-body terms are very small.

# Three-body Casimir Energy

The multiple scattering formulation has proved exceptionally useful in computing Casimir energies for complex configurations. It is usually presented in terms of potentials, where the potential stands in for the deviation of the permittivity from its vacuum value, for instance. Here, however, we wish from the outset to consider perfect conductors, so we wish to give the formulation entirely in terms of scattering matrices. In particular, we wish to analyze three-body effects.

# Quantum Vacuum Energy

The quantum vacuum energy is in general given by

$$E = \frac{i}{2} \text{Tr} \ln \mathbf{\Gamma} \mathbf{\Gamma}_0^{-1},$$

where  $\mathbf{\Gamma}_0$  is the free Green's dyadic, which, for a given frequency  $\omega$ , can be written as

$$\mathbf{\Gamma}_0(\mathbf{r}, \mathbf{r}') = (\mathbf{1}\omega^2 + \nabla\nabla)G(|\mathbf{r} - \mathbf{r}'|),$$

in terms of the Helmholtz Green's function

$$G(R) = \frac{e^{i|\omega|R}}{4\pi R}, \quad R = |\mathbf{r} - \mathbf{r}'|.$$



# Differential equation

The full Green's dyadic  $\Gamma$  satisfies the same differential equation as the free Green's dyadic,

$$\Gamma_0^{-1} \Gamma = \mathbf{1},$$

where

$$\Gamma_0^{-1} = \frac{1}{\omega^2} \nabla \times \nabla \times - \mathbf{1}.$$

Here we have adopted a matrix notation for both the tensor indices and the spatial coordinates, so

$$\mathbf{1} = \mathbf{1} \delta(\mathbf{r} - \mathbf{r}'),$$

where on the right  $\mathbf{1}$  refers to the tensor indices.

# Perfectly conducting boundaries

The conducting surfaces  $S$  appear through boundary conditions on the Green's dyadic,

$$\hat{\mathbf{n}} \times \boldsymbol{\Gamma} \Big|_S = 0,$$

where  $\hat{\mathbf{n}}$  is the outward normal to the surface at the point in question.

Now we may define the scattering operator  $\mathbf{T}$  by

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_0 \mathbf{T} \boldsymbol{\Gamma}_0,$$

so that

$$\mathbf{T} = -\boldsymbol{\Gamma}_0^{-1} + \boldsymbol{\Gamma}_0^{-1} \boldsymbol{\Gamma} \boldsymbol{\Gamma}_0^{-1}.$$

# Three bodies

Now we turn to the quantum interaction of three bodies. It seems easiest to start with the situation where the bodies may be described by potentials  $\mathbf{V}_i$ ,  $i = 1, 2, 3$ , and then write the result in a form in which only the  $\mathbf{T}$  operators appear, so it applies to the conducting boundary problem. The total potential is  $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3$ , and the vacuum energy is given by the trace-log of

$$\mathbf{\Gamma}\mathbf{\Gamma}_0^{-1} = (1 - \mathbf{\Gamma}_0\mathbf{V})^{-1},$$

or

$$E = -\frac{i}{2} \text{Tr} \ln(1 - \mathbf{\Gamma}_0\mathbf{V}).$$

# Isolating three-body terms

Now it is easy to see that

$$\begin{aligned} 1 - \Gamma_0(\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3) &= (1 - \Gamma_0\mathbf{V}_1 - \Gamma_0\mathbf{V}_2) \\ &\times \left[ 1 - (1 - \Gamma_1\mathbf{V}_2)^{-1}\Gamma_1\mathbf{V}_2\Gamma_1\mathbf{V}_3(1 - \Gamma_1\mathbf{V}_3)^{-1} \right] \\ &\times (1 - \Gamma_0\mathbf{V}_1)^{-1}(1 - \Gamma_0\mathbf{V}_1 - \Gamma_0\mathbf{V}_2). \end{aligned}$$

Here we have introduced the Green's dyadic belonging to potential  $i$  alone,

$$\Gamma_i = (1 - \Gamma_0\mathbf{V}_i)^{-1}\Gamma_0.$$

# Modified scattering operator

Now in the above expression the pre- and post-factors refer to only one- and two-body interactions (the latter referring to interactions between bodies 1 and 2, and 1 and 3, respectively), so the two-body interactions between 2 and 3, and three-body interactions are all contained in the quantity in square brackets. Now, in terms of the potential, the corresponding scattering operator is

$$\mathbf{T}_i = \mathbf{V}_i (\mathbf{1} - \mathbf{\Gamma}_0 \mathbf{V}_i)^{-1}.$$

Modified scattering operator:  $\tilde{\mathbf{T}} = \mathbf{T} \mathbf{\Gamma}_0$ .

# TGTG and Beyond

Using the cyclic property of the trace, we find the two and three body terms:

$$E_{23} = -\frac{i}{2} \text{Tr} \ln(\mathbf{1} - \tilde{\mathbf{T}}_2 \tilde{\mathbf{T}}_3), ,$$

which is the famous *TGTG* formula, and

$$E_{123} = -\frac{i}{2} \text{Tr} \ln \left( \mathbf{1} - \mathbf{X}_{23} \left[ \mathbf{X}_{21} \tilde{\mathbf{T}}_2 (\mathbf{1} + \tilde{\mathbf{T}}_1) \mathbf{X}_{31} \tilde{\mathbf{T}}_3 \right. \right. \\ \left. \left. \times (\mathbf{1} + \tilde{\mathbf{T}}_1) - \tilde{\mathbf{T}}_2 \tilde{\mathbf{T}}_3 \right] \right),$$

where  $\mathbf{X}_{ij} = (\mathbf{1} - \tilde{\mathbf{T}}_i \tilde{\mathbf{T}}_j)^{-1}$ .

# Cf. Shajesh and Schaden

This is not written in as symmetrical a form as in Shajesh and Schaden, <sup>a</sup> but is somewhat simpler, particular for the Casimir-Polder applications that follow, where body 1 represents the atom, so is treated weakly.

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<sup>a</sup>K. V. Shajesh and M. Schaden, Phys. Rev. A **83**, 125032 (2011).

# Atom between || conducting plates

Here we consider an anisotropically polarizable atom between parallel conducting plates, a geometry first considered by Barton.<sup>a</sup> From the Green's dyadic  $\Gamma$  for parallel plates, is inserted into the general Casimir-Polder formula

$$E_{CP} = - \int_{-\infty}^{\infty} d\zeta \operatorname{Tr} \alpha \cdot \Gamma, \quad \omega \rightarrow i\zeta,$$

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<sup>a</sup>G. Barton, Proc. Roy. Soc. London A **410**, 141 (1987).



# Full CP intreraction

The result of the straightforward calculation is, for one conducting plate at  $z = 0$ , one at  $z = a$ , and the atom at  $z = Z$ ,  $0 < Z < a$ ,

$$E = \frac{\alpha_{11} + \alpha_{22} - \alpha_{33}}{4\pi a^4} \zeta(4) - \frac{\text{tr } \boldsymbol{\alpha}}{8\pi a^4} \left[ \zeta\left(4, \frac{Z}{a}\right) + \zeta\left(4, 1 - \frac{Z}{a}\right) \right]$$

in terms of the Hurwitz zeta function.

# Isolation of 2- and 3-body effects

Here the two-body interactions between the atom and one or the other plate are isolated by extracting the parts singular as  $z \rightarrow a$  or  $z \rightarrow 0$ :

$$\zeta(4, Z/a) = \left(\frac{a}{Z}\right)^4 + \zeta(4, 1 + Z/a),$$

$$\zeta(4, 1 - Z/a) = \left(\frac{a}{a - Z}\right)^4 + \zeta(4, 2 - Z/a).$$

# 2- and 3-body terms

The total Casimir-Polder energy is the sum of two-body and three-body terms, (1 = atom)

$$E = E_{12} + E_{13} + E_{123},$$

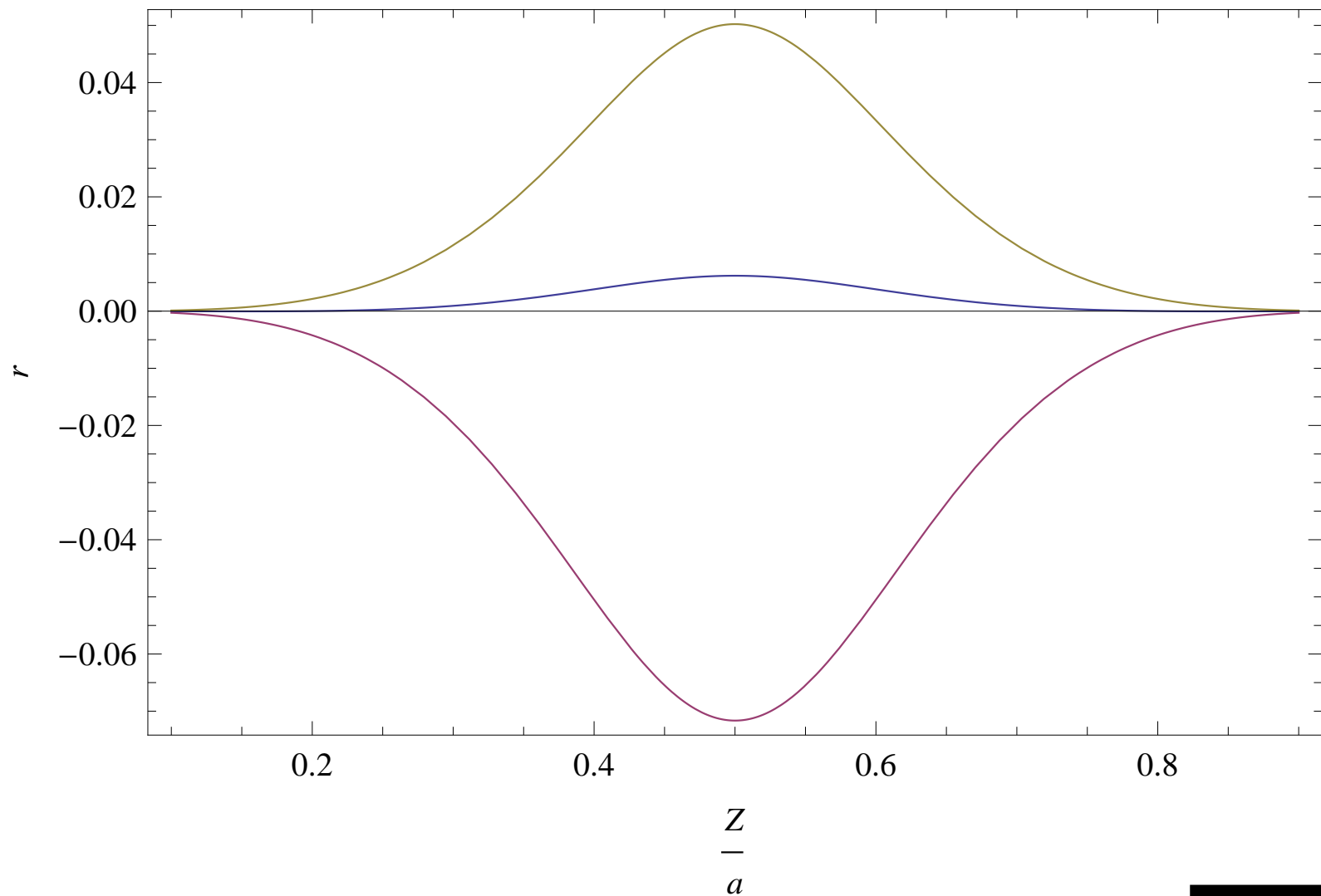
$$E_{12} = -\frac{\text{tr } \alpha}{8\pi Z^4}, \quad E_{13} = -\frac{\text{tr } \alpha}{8\pi (a - Z)^4},$$

$$E_{123} = \frac{\alpha_{11} + \alpha_{22} - \alpha_{33}}{4\pi a^4} \zeta(4) - \frac{\text{tr } \alpha}{8\pi a^4} \left[ \zeta \left( 4, 1 + \frac{Z}{a} \right) + \zeta \left( 4, 2 - \frac{Z}{a} \right) \right].$$

# 3-body interaction small

Note that the first term here is independent of  $Z$ , so it does not contribute to the Casimir-Polder force on the atom, but is a Casimir-Polder correction to the Casimir force between the plates. **The two-body energies overwhelmingly dominate the Casimir-Polder interaction**, as shown in the figure. For isotropic atoms, the largest three body correction is only 0.6% at the midpoint between the plates, where the energy is very small.

$$r = E_{123} / (E_{12} + E_{13})$$



# Multiple-scattering calculation

Since in more general situations we do not have an exact solution available, we want to calculate the three-body corrections using the multiple-scattering formula. For this purpose, we need to compute the scattering operators for the three bodies.

For the atom, this is easy:

$$\mathbf{T}_1(\mathbf{r}, \mathbf{r}') = \mathbf{V}_1(\mathbf{r}, \mathbf{r}') = 4\pi\alpha\delta(\mathbf{r} - \mathbf{R})\delta(\mathbf{r} - \mathbf{r}'),$$

where  $\mathbf{R} = (0, 0, Z)$  is the position of the atom.

# Free Green's dyadic

The free electromagnetic Green's dyadic can be written as

$$\mathbf{\Gamma}_0(\mathbf{r} - \mathbf{r}') = \int \frac{(d\mathbf{k}_\perp)}{(2\pi)^2} e^{i\mathbf{k}_\perp \cdot (\mathbf{r} - \mathbf{r}')_\perp} \gamma_0(z, z'),$$

where

$$\gamma_0(z, z') = (\mathbf{E} + \mathbf{H}) \frac{1}{2\kappa} e^{-\kappa|z-z'|},$$

with the usual abbreviation  $\kappa = \sqrt{k^2 + \zeta^2}$ .

# $\tilde{T}$ for atom

Here  $\mathbf{E}$  and  $\mathbf{H}$  are matrices corresponding to the transverse electric (TE) and transverse magnetic (TM) modes,

$$\frac{\mathbf{E}}{-\zeta^2} = \begin{pmatrix} s^2 & -cs & 0 \\ -cs & c^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{H} = \begin{pmatrix} c^2 \partial_z \partial_{z'} & cs \partial_z \partial_{z'} & ikc \partial_z \\ cs \partial_z \partial_{z'} & s^2 \partial_z \partial_{z'} & iks \partial_z \\ -ikc \partial_{z'} & -iks \partial_{z'} & k^2 \end{pmatrix}$$

here  $k^2 = \mathbf{k}_\perp^2$  and  $c$  ( $s$ ) is the cosine (sine) of the angle between the direction of  $\mathbf{k}_\perp$  and the  $x$ -axis,  $c = k_x/k$ ,  $s = k_y/k$ .



# $\tilde{T}$ for atom

Thus the reduced modified scattering operator for the atom is

$$\tilde{\mathbf{t}}_1(z, z') = 4\pi\alpha\delta(z - Z)(\mathbf{E} + \mathbf{H})(Z, z')\frac{1}{2\kappa}e^{-\kappa|Z-z'|}.$$

# Properties of E and H

The following composition properties of the  $\mathbf{E}$  and  $\mathbf{H}$  operators are easily checked:

$$\mathbf{EH} = 0,$$

$$\mathbf{EE} = -\zeta^2 \mathbf{E},$$

$$\mathbf{H}(z, z')\mathbf{H}(z'', z''') = (k^2 + \partial_{z'}\partial_{z''})\mathbf{H}(z, z''').$$

# Reduced Green's dyadic for plate

For a single plate, say a conducting plate 2 at  $z = 0$ , we have the reduced Green's dyadic in the form

$$\gamma = \mathbf{E}g^E + \mathbf{H}g^H,$$

where

$$g^{E,H}(z, z') = g_0(z, z') \pm \frac{1}{2\kappa} e^{-\kappa(|z|+|z'|)},$$

$$g_0(z, z') = \frac{1}{2\kappa} e^{-\kappa|z-z'|}.$$

# $\tilde{T}$ for plate

Then the modified scattering operator is

$$\tilde{\mathbf{t}}_2(z, z') = \gamma_0^{-1}(\gamma - \gamma_0)(z, z').$$

This is evaluated by using

$$\left( -\frac{d^2}{dz^2} + \kappa^2 \right) e^{-\kappa|z|} = 2\kappa\delta(z).$$

Thus the scattering operator for a conducting plate at  $z = 0$  is

$$\tilde{\mathbf{t}}_2(z, z') = \frac{1}{\zeta^2} (\mathbf{E} - \mathbf{H})(z, z') \delta(z) e^{-\kappa|z'|}.$$

# Atom-plate interaction

Let us check this by computing the CP interaction between an atom and a single conducting plate,

$$\begin{aligned} E_{12} &= \frac{i}{2} \text{Tr } \tilde{\mathbf{T}}_1 \tilde{\mathbf{T}}_2 \\ &= -\frac{1}{2} \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \int \frac{(d\mathbf{k}_{\perp})}{(2\pi)^2} 4\pi \text{tr } \boldsymbol{\alpha} \int dz dz' \delta(z - Z) \\ &\quad \times (\mathbf{E} + \mathbf{H})(z, z') \frac{e^{-\kappa(z-z')}}{2\kappa} \frac{(\mathbf{E} - \mathbf{H})(z', z)}{\zeta^2} \delta(z') e^{-\kappa z}. \end{aligned}$$

Because  $k^2 + \partial_{z'} \partial_{z''} \rightarrow -\zeta^2$  we have

$$(\mathbf{E} + \mathbf{H})(\mathbf{E} - \mathbf{H}) = -\zeta^2 (\mathbf{E} - \mathbf{H}).$$

# CP interaction

Now when we carry out the integration over transverse momentum the off-diagonal terms integrate away, and  $s^2, c^2 \rightarrow \frac{1}{2}$ ,  $cs \rightarrow 0$ .

$$E_{12} = \frac{1}{2\pi} \int_0^\infty d\kappa \kappa^2 d\cos\theta \frac{e^{-2\kappa Z}}{2\kappa} \text{tr } \alpha \left[ \begin{pmatrix} -\frac{\zeta^2}{2} & 0 & 0 \\ 0 & -\frac{\zeta^2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right. \\ \left. - \begin{pmatrix} \frac{\kappa^2}{2} & 0 & 0 \\ 0 & \frac{\kappa^2}{2} & 0 \\ 0 & 0 & k^2 \end{pmatrix} \right].$$

# Usual 2-body CP energy

Here  $\theta$  is the polar angle in the three-dimensional space  $(k_x, k_y, \zeta)$ , so when that angle is integrated over, we have the replacements

$$\zeta^2 \rightarrow \frac{2}{3}\kappa^2, \quad k^2 \rightarrow \frac{4}{3}\kappa^2, \quad \kappa^2 \rightarrow 2\kappa^2,$$

with the expected result:

$$E_{12} = -\frac{\text{tr } \alpha}{8\pi Z^4}.$$

# $E_{CP}$ for atom and two plates

The three-body interaction is worked out in a very similar manner. We start by simplifying the multiple-scattering formula for the case when there is only one interaction with the atom, since that coupling is always weak:

$$E_{123} = \frac{i}{2} \text{Tr} X_{23} \left( \tilde{T}_2 \tilde{T}_1 \tilde{T}_2 \tilde{T}_3 + \tilde{T}_2 \tilde{T}_3 \tilde{T}_1 \tilde{T}_3 \right. \\ \left. + \tilde{T}_2 \tilde{T}_1 \tilde{T}_3 + \tilde{T}_2 \tilde{T}_3 \tilde{T}_1 \right).$$



# TE contribution

Let's look at the  $\mathbf{E}$  and  $\mathbf{H}$  parts separately. For the TE part,

$$\mathbf{X}_{23}\mathbf{E} = \mathbf{E} \frac{1}{1 - e^{-2\kappa a}} \delta(z),$$

so

$$E_{123}^{\text{TE}} = \frac{1}{4\pi^2} \int d\zeta (d\mathbf{k}_{\perp}) \text{tr} \boldsymbol{\alpha} \left( -\frac{\mathbf{E}}{\zeta^2} \right) \frac{e^{-2\kappa a}}{1 - e^{-2\kappa a}} \left( -\frac{\zeta^2}{2\kappa} \right) \\ \times \left[ e^{-2\kappa Z} + e^{-2\kappa(a-Z)} - 2 \right],$$

# TE contribution continued

where integrating over the directions of  $\mathbf{k}_\perp$  gives for the trace

$$\text{tr } \alpha \left( -\frac{\mathbf{E}}{\zeta^2} \right) \rightarrow \frac{1}{2}(\alpha_{11} + \alpha_{22}).$$

Thus the TE contribution is

$$\begin{aligned} E_{123}^{\text{TE}} &= -\frac{\alpha_{11} + \alpha_{22}}{12\pi} \int_0^\infty d\kappa \kappa^3 \frac{[-2 + e^{-2\kappa Z} + e^{-2\kappa(a-Z)}]}{e^{2\kappa a} - 1} \\ &= \frac{\alpha_{11} + \alpha_{22}}{32\pi a^4} \left[ 2\zeta(4) - \zeta\left(4, 1 + \frac{Z}{a}\right) - \zeta\left(4, 2 - \frac{Z}{a}\right) \right] \end{aligned}$$

# TM contribution

The TM contribution is similarly worked out, with the result  $E_{123}^{\text{TM}} =$

$$\begin{aligned} &= \frac{1}{2\pi} \int_0^\infty \frac{d\kappa \kappa^3}{e^{2\kappa a} - 1} \left\{ \frac{\alpha_{11} + \alpha_{22}}{2} \left[ 2 - e^{-2\kappa Z} - e^{2\kappa(a-Z)} \right] \right. \\ &\quad \left. + \frac{2}{3} \alpha_{33} \left[ e^{-2\kappa Z} + e^{-2\kappa(a-Z)} + 2 \right] \right\} \\ &= \frac{3(\alpha_{11} + \alpha_{22})}{32\pi a^4} \left[ 2\zeta(4) - \zeta\left(4, 1 + \frac{Z}{a}\right) - \zeta\left(4, 2 - \frac{Z}{a}\right) \right] \\ &\quad - \frac{\alpha_{33}}{8\pi a^4} \left[ 2\zeta(4) + \zeta(4, 1 + Z/a) + \zeta(4, 2 - Z/a) \right]. \end{aligned}$$

Adding this to the TE contribution gives  $E_{123}$ .

# CP force due to facing wedges

Last year we showed results for the CP interaction between an anisotropic atom and a single wedge. More interesting is the case of two facing wedges which includes the case of two half-planes separated by a rectangular gap. We expect that the three-body correction to the known two-body forces are small. **The calculation is in progress.**

# Two-body atom-wedge ( $p = \pi/\Omega$ )

$$\Gamma(\mathbf{r}, \mathbf{r}') = 2p \sum_{m=0}^{\infty}{}' \int \frac{dk}{2\pi} \left[ -\mathcal{M} \mathcal{M}'^* (\nabla_{\perp}^2 - k^2) \right. \\ \times \frac{1}{\omega^2} F_{mp}(\rho, \rho') \frac{\cos mp\theta \cos mp\theta'}{\pi} e^{ik(y-y')} \\ \left. + \mathcal{N} \mathcal{N}'^* \frac{1}{\omega} G_{mp}(\rho, \rho') \frac{\sin mp\theta \sin mp\theta'}{\pi} e^{ik(y-y')} \right]$$

$$\mathcal{M} = \hat{\rho} \frac{\partial}{\rho \partial \theta} - \hat{\theta} \frac{\partial}{\partial \rho},$$

$$\mathcal{N} = ik \left( \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\theta} \frac{\partial}{\rho \partial \theta} \right) - \hat{y} \nabla_{\perp}^2.$$

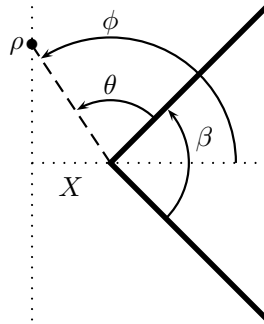
# Free reduced Green's functions

In this situation, the boundaries are entirely in planes of constant  $\theta$ , so the radial Green's functions are equal to the free Green's function

$$\frac{1}{\omega^2} F_{mp}(\rho, \rho') = \frac{1}{\omega} G_{mp}(\rho, \rho') = -\frac{i\pi}{2\lambda^2} J_{mp}(\lambda\rho_{<}) H_{mp}^{(1)}(\lambda\rho_{>})$$

We will immediately make the Euclidean rotation,  $\omega \rightarrow i\zeta$ , where  $\lambda \rightarrow i\kappa$ ,  $\kappa^2 = \zeta^2 + k^2$ , so the free Green's functions become  $-\kappa^{-2} I_{mp}(\kappa\rho_{<}) K_{mp}(\kappa\rho_{>})$ .

# Repulsion by a wedge



We want the  $z$  axis to be perpendicular to the symmetry axis of the wedge where, as before,  $\theta$  is the angle relative to the top surface of the We consider a strongly anisotropic atom, with only  $\alpha_{zz}$  significant, to the left of a wedge of opening angle  $\beta = 2\pi - \Omega$ . How does repulsion depends on  $\beta$ ?

# Dependence on wedge angle

Write for an atom on the line  $x = -X$

$$U_{\text{CP}}^{zz} = -\frac{\alpha_{zz}(0)}{8\pi X^4} V(\phi),$$

where

$$V(\phi) = \cos^4 \phi \left[ \frac{p^4}{\sin^4 \frac{\pi \phi - \beta/2}{2}} - \frac{2 p^2 (p^2 - 1)}{3 \sin^2 \frac{\pi \phi - \beta/2}{2}} + \frac{1}{45} (p^2 - 1)(p^2 + 11) \cos 2\phi \right].$$



# Vanishing of energy?

At the point of closest approach,

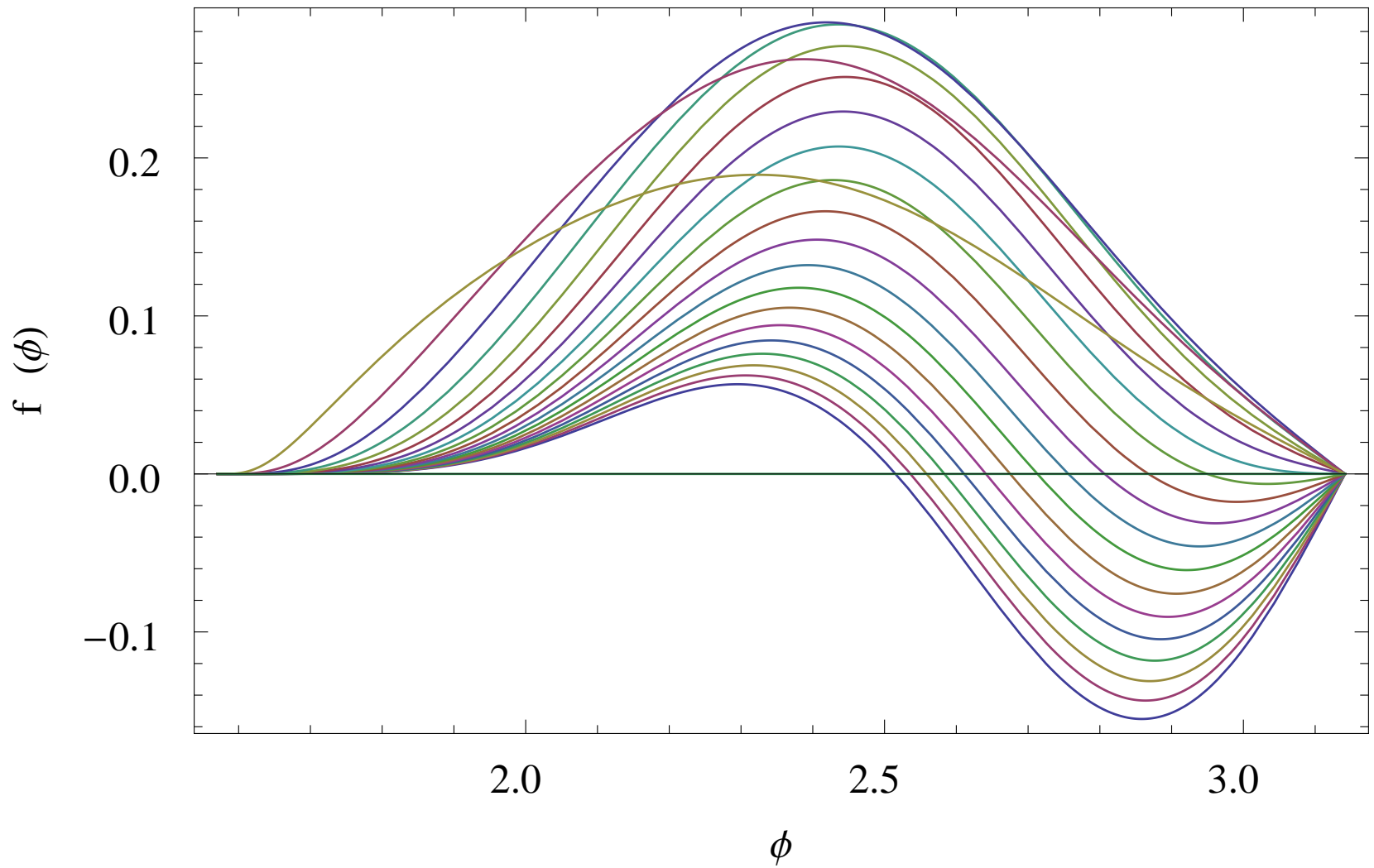
$$V(\pi) = \frac{1}{45}(4p^2 - 1)(4p^2 + 11),$$

so the potential vanishes at that point only for the half-plane case,  $p = 1/2$ .

$$F_z = -\frac{\alpha_{zz}}{8\pi} \frac{1}{X^5} \cos^2 \phi \frac{\partial V(\phi)}{\partial \phi}.$$

The figure shows the force as a function of  $\phi$  for fixed  $X$ . It will be seen that the force has a repulsive region for angles close enough to the apex of the wedge, provided that the wedge angle is not too large. The critical wedge angle is actually rather large,  $\beta_c = 1.87795$ , or about  $108^\circ$ . For larger angles, the  $z$ -component of the force exhibits only attraction.

# $F_z$ for $\beta \in [0, \pi]$



# 3-body forces due to facing wedges

We can use the above formalism to describe the three-body correction to the CP force on an atom passing along the symmetry line between facing, nonoverlapping, wedges. The two-body effect would be exactly twice that due to one wedge. The three-body correction can be computed from the  $\tilde{T}$  matrix for a single wedge, which is

# $\tilde{T}$ for wedge

$$\begin{aligned} \tilde{T}(\mathbf{r}, \mathbf{r}') = & 2p \sum_{m=0}^{\infty} ' \int dk 2\pi \left[ \frac{\mathbf{E}}{\zeta^2} \frac{1}{\rho^2} \left( \frac{1}{\kappa^2} I_{mp}(\kappa\rho_{<}) K_{mp}(\kappa\rho_{>}) \right. \right. \\ & \times \frac{1}{\pi} [\delta'(\theta) - (-1)^m \delta'(\theta - \Omega)] \cos p\theta' \\ & + \frac{\mathbf{H}}{\zeta^2} \frac{1}{\rho^2} \left( -\frac{1}{\kappa^2} I_{mp}(\kappa\rho_{<}) K_{mp}(\kappa\rho_{>}) \right) \\ & \left. \left. \times \frac{mp}{\pi} [\delta(\theta) - (-1)^m \delta(\theta - \Omega)] \sin p\theta' \right] e^{ik(z-z')}. \end{aligned}$$

Here  $\mathbf{E} = \mathcal{M} \mathcal{M}'^* (\nabla_{\perp}^2 - k^2)$ ,  $\mathbf{H} = \mathcal{N} \mathcal{N}'^*$

# Conclusion

- Casimir and Casimir-Polder repulsion can occur in anisotropic geometries.
- Three-body effects are expected, generically, to be small.
- This is demonstrated in a rather trivial example, the interaction of an atom with two parallel planes.
- Thus a gap between two conducting wedges, or two conducting half planes, should give rise to repulsion on a sufficiently anisotropic atom, since a single wedge leads to such repulsion.