Pointwise Bounds and Blow-up for Nonlinear Fractional Parabolic Inequalities

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Abstract

We investigate pointwise upper bounds for nonnegative solution \( u(x,t) \) of the nonlinear initial value problem

\[
0 \leq (\partial_t - \Delta)^\alpha u \leq u^\lambda \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}, \quad n \geq 1, \quad (1.1)
\]

\[
u = 0 \quad \text{in} \quad \mathbb{R}^n \times (-\infty,0) \quad (1.2)
\]

where \( \lambda \) and \( \alpha \) are positive constants. To do this we first give a definition—tailored for our study of (0.1), (0.2)—of fractional powers of the heat operator \((\partial_t - \Delta)^\alpha : Y \to X\) where \(X\) and \(Y\) are linear spaces whose elements are real valued functions on \(\mathbb{R}^n \times \mathbb{R}\) and \(0 < \alpha < \alpha_0\) for some \(\alpha_0\) which depends on \(n, X\) and \(Y\).

We then obtain, when they exist, optimal pointwise upper bounds on \(\mathbb{R}^n \times (0, \infty)\) for nonnegative solutions \(u \in Y\) of the initial value problem (0.1), (0.2) with particular emphasis on those bounds as \(t \to 0^+\) and as \(t \to \infty\).

2010 Mathematics Subject Classification. 35B09, 35B33, 35B44, 35B45, 35K58, 35R11, 35R45.

Keywords. Blow-up, Pointwise bounds, Fractional heat operator, Parabolic.

1 Introduction

In this paper we study pointwise upper bounds for nonnegative solutions \(u(x,t)\) of the nonlinear inequalities

\[
0 \leq (\partial_t - \Delta)^\alpha u \leq u^\lambda \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}, \quad n \geq 1, \quad (1.1)
\]

satisfying the initial condition

\[
u = 0 \quad \text{in} \quad \mathbb{R}^n \times (-\infty,0) \quad (1.2)
\]

where \(\lambda\) and \(\alpha\) are positive constants.

To do this, we first give in Section 2 a definition—appropriate for our analysis of the initial value problem (1.1), (1.2)—of fractional powers of the heat operator

\[
(\partial_t - \Delta)^\alpha : Y \to X \quad (1.3)
\]

where \(\Delta\) is the Laplacian with respect to \(x \in \mathbb{R}^n\), \(X\) and \(Y\) are linear spaces whose elements are real valued functions on \(\mathbb{R}^n \times \mathbb{R}\), and \(0 < \alpha < \alpha_0\) for some \(\alpha_0 > 0\) which depends on \(n, X\) and \(Y\).

With the definition of (1.3) in hand, we obtain, when they exist, optimal pointwise upper bounds on \(\mathbb{R}^n \times (0, \infty)\) for nonnegative solutions \(u \in Y\) of the initial value problem (1.1), (1.2) with
particular emphasis on these bounds as $t \to 0^+$ and as $t \to \infty$. These results are stated in Section 3 and proved in Section 8.

Since the operator (1.3) is nonlocal, we must require the initial condition (1.2) to hold in $\mathbb{R}^n \times (-\infty, 0)$ (not just in $\mathbb{R}^n \times \{0\}$) and nonnegative solutions of (1.1), (1.2) may not tend pointwise to zero as $t \to 0^+$ (see Theorem 3.5) even though they satisfy the initial condition (1.2).

Of course any estimates we obtain for nonnegative solutions of (1.1), (1.2) also hold for nonnegative solutions of the initial value problem consisting of (1.2) and the equation

$$(\partial_t - \Delta)^\alpha u = u^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R}.$$

The operator (1.3) is a fully fractional heat operator as opposed to time fractional heat operators in which the fractional derivatives are only with respect to $t$, and space fractional heat operators, in which the fractional derivatives are only with respect to $x$.

Some recent results for nonlinear PDEs containing time (resp. space) fractional heat operators can be found in [3, 4, 13, 14, 18, 25, 29] (resp. [1, 2, 6, 7, 8, 9, 10, 12, 15, 19, 26, 27, 28]). We know of no results for nonlinear PDEs containing the fully fractional heat operator (1.3). However results for linear PDEs containing (1.3), including in particular

$$(\partial_t - \Delta)^\alpha u = f,$$

where $f$ is a given function, can be found in [5, 17, 21, 24].

## 2 Definition and properties of fully fractional heat operators

In this section we give a well-motivated definition of the fully fractional heat operator (1.3), suitable for our study of the initial value problem (1.1), (1.2), and then give some of its properties.

Some of the material in this section is inspired by—and can be viewed as the parabolic analog of—the material in [23, Sec. 5.1] concerning the fractional Laplacian.

Since for functions $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $n \geq 1$, which are sufficiently smooth and small at infinity we have

$$(\partial_t - \Delta)\hat{u}(y,s) = (|y|^2 - is)\hat{u}(y,s),$$

where $\hat{u}$ is the Fourier transform operator on $\mathbb{R}^n \times \mathbb{R}$ given by

$$\hat{u}(y,s) = \int_{\mathbb{R}^n \times \mathbb{R}} e^{i(y,s) \cdot (x,t)} u(x,t) \, dx \, dt,$$

the fractional heat operator $(\partial_t - \Delta)^\alpha$, $\alpha > 0$, is formally defined in [22, Chapter 2] by

$$(\partial_t - \Delta)^\alpha \hat{u}(y,s) = (|y|^2 - is)^\alpha \hat{u}(y,s).$$

(2.1)

If $f = (\partial_t - \Delta)^\alpha u$ then from (2.1) and the fact (see [22, Theorem 2.2] and Theorem 2.1(i) below) that

$$\hat{\Phi}_\alpha(y,s) = (|y|^2 - is)^{-\alpha} \quad \text{for } 0 < \alpha < (n + 2)/2$$

in the sense of tempered distributions where

$$\Phi_\alpha(x,t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)} \chi_{(0,\infty)}(t),$$

we formally get

$$\hat{u} = \hat{\Phi}_\alpha \hat{f}.$$
Hence by the convolution theorem we formally find that

\[ u = J_\alpha f := \Phi_\alpha * f \]  

(2.3)

where \( * \) is the convolution operation in \( \mathbb{R}^n \times \mathbb{R} \). Since \( \Phi_\alpha(x, t) = 0 \) for \( t \leq 0 \) we have

\[ J_\alpha f(x, t) = \int_{\mathbb{R}^n \times (-\infty, t)} \Phi_\alpha(x - \xi, t - \tau) f(\xi, \tau) d\xi d\tau. \]  

(2.4)

By part (ii) of the following theorem, equations (2.1) and (2.3) are equivalent in the sense that

\[ (J_\alpha f)^\wedge = (|y|^2 - is)^{-\alpha} \hat{f} \]  

for \( f \in L^1(\mathbb{R}^n \times \mathbb{R}) \) and \( 0 < \alpha < (n + 2)/2 \) in the sense of tempered distributions.

**Theorem 2.1.** Suppose \( 0 < \alpha < (n + 2)/2 \).

(i) The Fourier transform of \( \Phi_\alpha(x, t) \) is the function \((|y|^2 - is)^{-\alpha}\) in the sense that

\[ \int_{\mathbb{R}^n \times \mathbb{R}} \Phi_\alpha(x, t) \hat{\varphi}(x, t) dx dt = \int_{\mathbb{R}^n \times \mathbb{R}} (|y|^2 - is)^{-\alpha} \varphi(y, s) dy ds \]

for all \( \varphi \in S \) where \( S \) is the Schwarz class of rapidly decreasing functions.

(ii) The identity \((J_\alpha f)^\wedge (y, t) = (|y|^2 - is)^{-\alpha} \hat{f}(y, s)\) holds in the sense that

\[ \int_{\mathbb{R}^n \times \mathbb{R}} J_\alpha f(x, t) \hat{g}(x, t) dx dt = \int_{\mathbb{R}^n \times \mathbb{R}} (|y|^2 - is)^{-\alpha} \hat{f}(y, s) g(y, s) dy ds \]  

(2.5)

for all \( f \in L^1(\mathbb{R}^n \times \mathbb{R}) \) and all \( g \in S \).

Motivated by these formal calculations, we will now define the operator \((\partial_t - \Delta)^{\alpha}\) as the inverse of a linear operator

\[ J_\alpha : X \to Y \]  

(2.6)

where \( J_\alpha \) is defined by (2.4) and (2.2) and \( X \) and \( Y \) are linear spaces whose elements are functions \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) such that the operator (2.6) has the following properties:

(P1) it makes sense,

(P2) it is one-to-one and onto, and

(P3) if \( u = J_\alpha f \) then \( f = 0 \) in \( \mathbb{R}^n \times (-\infty, 0) \) if and only if \( u = 0 \) in \( \mathbb{R}^n \times (-\infty, 0) \).

Property (P3) will be needed to handle the initial condition (1.2). The domain of \( J_\alpha \) is usually taken to be \( L^p(\mathbb{R}^n \times \mathbb{R}) \), \( 1 \leq p < \frac{2(n+2)}{2n} \) (see [21, Section 9.2]). However since the region of integration for the integral (2.4) is not \( \mathbb{R}^n \times \mathbb{R} \) but rather \( \mathbb{R}^n \times (-\infty, t) \), we see that more natural and less restrictive choices for the domain and range of \( J_\alpha \) are

\[ X^p := \bigcap_{T \in \mathbb{R}} L^p(\mathbb{R}^n \times \mathbb{R}_T) \]  

(2.7)

\[ Y^\alpha_p := J_\alpha(X^p) \]  

(2.8)
respectively, where \( \mathbb{R}_T = (-\infty, T) \). By (2.7) we mean \( X^p \) is the set of all measurable functions \( f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
\| f \|_{L^p(\mathbb{R}^n \times \mathbb{R}_T)} < \infty \quad \text{for all } T \in \mathbb{R}.
\]

The notation in (2.7) should be interpreted similarly elsewhere in this paper.

According to the following two theorems the formal operator

\[
J_\alpha : X^p \rightarrow Y^p_\alpha,
\]

where \( X^p \) and \( Y^p_\alpha \) are defined in (2.7) and (2.8), satisfies properties (P1)–(P3) provided either

\[
\left( p > 1 \text{ and } 0 < \alpha < \frac{n + 2}{2p} \right) \quad \text{or} \quad \left( p = 1 \text{ and } 0 < \alpha \leq \frac{n + 2}{2p} \right).
\]  

(2.10)

When \( p \) and \( \alpha \) satisfy (2.10), part (i) of the following theorem shows that the operator (2.9) satisfies (P1) and parts (ii) and (iii) give some of its properties.

**Theorem 2.2.** Suppose \( p \) and \( \alpha \) are real numbers satisfying (2.10) and \( f \in X^p \). Then

(i) \( J_\alpha f, J_\alpha |f| \in L^p_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}) \) and

(ii) \( J_\beta (J_\gamma f) = J_\alpha f \) in \( L^p_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}) \) whenever \( \beta > 0, \gamma > 0, \text{ and } \beta + \gamma = \alpha \).

If in addition, \( \alpha > 1 \) then

(iii) \( H J_\alpha f = J_{\alpha-1} f \) in \( \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}) \) where \( H = \partial_t - \Delta \) is the heat operator.

**Remark 2.1.** Theorem 2.2(i) can be improved to \( J_\alpha f \in L^q_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}) \) when

\[
1 < p < \frac{n + 2}{2\alpha} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{n + 2}.
\]

This can be seen by applying Gopala Rao [11, Theorem 3.1] to the function \( f_T \) defined in the proof of Theorem 2.2 in Section 6.

According to the following theorem, if \( p \) and \( \alpha \) satisfy (2.10) then the operator (2.9) satisfies properties (P2) and (P3) where \( X^p \) and \( Y^p_\alpha \) are defined by (2.7) and (2.8).

**Theorem 2.3.** Suppose \( p \) and \( \alpha \) are real numbers satisfying (2.10). Then

(i) the operator (2.9) is one-to-one and onto, and

(ii) if

\[
f \in X^p \text{ and } T \in \mathbb{R}
\]

then

\[
f |_{\mathbb{R}^n \times \mathbb{R}_T} = 0 \quad \text{if and only if} \quad (J_\alpha f) |_{\mathbb{R}^n \times \mathbb{R}_T} = 0.
\]

By the results in this section, the following definition is natural and makes sense.

**Definition 2.1.** Suppose \( p \) and \( \alpha \) are real numbers satisfying (2.10) and \( X^p \) and \( Y^p_\alpha \) are defined by (2.7) and (2.8). Then the operator

\[
(\partial_t - \Delta)^\alpha : Y^p_\alpha \rightarrow X^p
\]

is defined to be the inverse of the operator (2.9).

**Remark 2.2.** The functions \( \mu_T : X^p \rightarrow \mathbb{R}, \ T \in \mathbb{R}, \) defined by \( \mu_T(f) = \| f \|_{L^p(\mathbb{R}^n \times \mathbb{R}_T)} \), form a separating family of seminorms on \( X^p \) which turns \( X^p \) into a locally convex topological vector space (see for example [20, Theorem 1.37]). Thus assuming (2.10) and defining a subset \( O' \) of \( Y^p_\alpha \) to be open if \( O' = J_\alpha(O) \) for some open set \( O \subset X^p \), we see by Theorem 2.3(i) that \( Y^p_\alpha \) is also a locally convex topological vector space and the operator (2.12) is a homeomorphism.
3 Results for fully fractional initial value problems

In this section we state our results concerning pointwise bounds for nonnegative solutions

\( u \in Y_\alpha^p \quad (3.1) \)

of the fully fractional initial value problem

\[
0 \leq (\partial_t - \Delta)^\alpha u \leq u^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \ n \geq 1, \quad (3.2)
\]

\[ u = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0) \quad (3.3) \]

where \( \lambda > 0 \) and, as in the Definition 2.1 of the operator (2.12), \( \alpha \) and \( p \) satisfy (2.10).

**Remark 3.1.** If \( \alpha \) and \( p \) satisfy (2.10) and \( u \) satisfies (3.1) and the first inequality in (3.2) then

\[ f := (\partial_t - \Delta)^\alpha u \geq 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R} \]

and hence \( u = J_\alpha f \geq 0 \) in \( \mathbb{R}^n \times \mathbb{R} \) by (2.4). Thus the assumption that \( u \) be nonnegative can be omitted when studying (3.1)–(3.3).

In order to state our results we first note that for each fixed \( p \geq 1 \) the open first quadrant of the \( \lambda \alpha \)-plane is the union of the following pairwise disjoint sets.

\[ A = \{ (\lambda, \alpha) : \lambda \geq 1 \text{ and } \alpha > \frac{n+2}{2p} \left( 1 - \frac{1}{\lambda} \right) \} \]

\[ B = \{ (\lambda, \alpha) : 0 < \lambda < 1 \text{ and } \alpha > 0 \} \]

\[ C = \{ (\lambda, \alpha) : \lambda > 1 \text{ and } 0 < \alpha < \frac{n+2}{2p} \left( 1 - \frac{1}{\lambda} \right) \} \]

\[ D = \{ (\lambda, \alpha) : \lambda > 1 \text{ and } \alpha = \frac{n+2}{2p} \left( 1 - \frac{1}{\lambda} \right) \} . \]

Note that \( A, B, \) and \( C \) are two dimensional regions in the \( \lambda \alpha \)-plane whereas \( D \) is the curve separating \( A \) and \( C \). (See Figure 1.) Our results in this section deal with solutions of (3.1)–(3.3) when \( (\lambda, \alpha) \) is in \( A, B, \) or \( C \). We have no results when \( (\lambda, \alpha) \in D \).

The following theorem deals with the case that \( (\lambda, \alpha) \in A \).

**Theorem 3.1.** Suppose \( \alpha \) and \( p \) satisfy (2.10), \( (\lambda, \alpha) \in A \), and \( u \) satisfies (3.1)–(3.3). Then

\[ u = (\partial_t - \Delta)^\alpha u = 0 \quad \text{almost everywhere in } \mathbb{R}^n \times \mathbb{R}. \]

The following three theorems deal with the case \( (\lambda, \alpha) \in B \).

**Theorem 3.2.** Suppose \( \alpha \) and \( p \) satisfy (2.10), \( (\lambda, \alpha) \in B \), and \( u \) satisfies (3.1)–(3.3). Then for all \( T > 0 \) we have

\[ \|u\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leq (MT^\alpha)^{\frac{1}{1-\lambda}} \quad (3.4) \]

and

\[ \|(\partial_t - \Delta)^\alpha u\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leq (MT^\alpha)^{\frac{\lambda}{1-\lambda}} \quad (3.5) \]

where

\[ M = M(\alpha, \lambda) = \frac{\Gamma(\frac{\alpha\lambda}{1-\lambda} + 1)}{\Gamma(\alpha + \frac{\alpha\lambda}{1-\lambda} + 1)} \quad (3.6) \]

where \( \Gamma \) is the Gamma function.
By the following theorem the bounds (3.4) and (3.5) in Theorem 3.2 are optimal.

**Theorem 3.3.** Suppose \( \alpha \) and \( p \) satisfy (2.10), \( (\lambda, \alpha) \in B, T > 0, \) and \( N < M \) where \( M \) is given by (3.6). Then there exists a solution \( u \in Y^p_\alpha \cap C(\mathbb{R}^n \times \mathbb{R}) \)

of (3.2), (3.3) such that

\[
(\partial_t - \Delta)^\alpha u \in L^p(\mathbb{R}^n \times \mathbb{R}) \cap C(\mathbb{R}^n \times \mathbb{R}),
\]

\[
u(0, t) \geq (N t^\alpha)^{\frac{1}{1-\lambda}} \quad \text{for} \ 0 < t < T
\]

and

\[
(\partial_t - \Delta)^\alpha u(0, t) = (N t^\alpha)^{\frac{1}{1-\lambda}} \quad \text{for} \ 0 < t < T.
\]

Although the estimates (3.4) and (3.5) are optimal there still remains the question as to whether there is a single solution which has the same size as these estimates as \( t \to \infty \). By the following theorem there is such a solution.

**Theorem 3.4.** Suppose \( \alpha \) and \( p \) satisfy (2.10) and \( (\lambda, \alpha) \in B \). Then there exists \( N > 0 \) and \( u \in Y^p_\alpha \) satisfying (3.2), (3.3) such that

\[
u(x, t) \geq (N t^\alpha)^{\frac{1}{1-\lambda}} \quad \text{for} \ (x, t) \in \Omega
\]

and

\[
(\partial_t - \Delta)^\alpha u(x, t) \geq (N t^\alpha)^{\frac{1}{1-\lambda}} \quad \text{for} \ (x, t) \in \Omega
\]

where \( \Omega = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |x|^2 < t\} \).

According to the following theorem, if \( (\lambda, \alpha) \in C \) then there exist bounds as \( t \to 0^+ \) for solutions of (3.1)–(3.3) in neither the pointwise (i.e. \( L^\infty \)) sense nor in the \( L^q \) sense when \( q > p \).

Moreover by Theorem 3.6 the same is true as \( t \to \infty \) provided \( q \in [q_0, \infty] \) for some \( q_0 = q_0(n, \alpha, \lambda) > p \).
Theorem 3.5. Suppose $\alpha$ and $p$ satisfy (2.10)
\[ (\lambda, \alpha) \in C \quad \text{and} \quad q \in (p, \infty). \]
Then there exists a solution $u \in Y^p_\alpha$ of (3.2), (3.3) and a sequence $\{t_j\} \subset (0, 1)$ such that
\[ \lim_{j \to \infty} t_j = 0 \]
and
\[ \|u^\lambda\|_{L^q(R^j)} = \| (\partial_t - \Delta)^\alpha u \|_{L^q(R^j)} = \infty \quad \text{for } j = 1, 2, \ldots, \]
where
\[ R_j = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |x| < \sqrt{t_j} \text{ and } t_j < t < 2t_j\}. \] (3.7)

Theorem 3.6. Suppose $\alpha$ and $p$ satisfy (2.10),
\[ (\lambda, \alpha) \in C \quad \text{and} \quad q \in \left[ \frac{n + 2}{2\alpha} \left( 1 - \frac{1}{\lambda} \right), \infty \right]. \]
Then there exists a solution $u \in Y^p_\alpha$ of (3.2), (3.3) and a sequence $\{t_j\} \subset (1, \infty)$ such that
\[ \lim_{j \to \infty} t_j = \infty \]
and
\[ \|u^\lambda\|_{L^q(R^j)} = \| (\partial_t - \Delta)^\alpha u \|_{L^q(R^j)} = \infty \quad \text{for } j = 1, 2, \ldots, \]
where $R_j$ is given in (3.7).

4 $J_\alpha$ version of fully fractional initial value problems

In order to prove our results stated in Section 3, we will first reformulate them in terms of the inverse $J_\alpha$ of the fractional heat operator (2.12) as follows.

Suppose that $\lambda > 0$ and, as assumed in Definition 2.1 and Theorems 3.1–3.6, that $p$ and $\alpha$ satisfy (2.10). Then, by Theorem 2.3, $u$ satisfies (3.1)–(3.3) if and only if $f := (\partial_t - \Delta)^\alpha u$ satisfies
\[ f \in X^p \] \hspace{1cm} (4.1)
\[ 0 \leq f \leq (J_\alpha f)^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R} \] \hspace{1cm} (4.2)
\[ f = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0). \] \hspace{1cm} (4.3)

Thus the two problems (3.1)–(3.3) and (4.1)–(4.3) are equivalent under the transformation $u = J_\alpha f$ when $p$ and $\alpha$ satisfy (2.10). This restriction on $p$ and $\alpha$ was imposed so that $J_\alpha f$ would be defined pointwise in $\mathbb{R}^n \times \mathbb{R}$ for all $f \in X^p$. If $p \geq 1$ and $\alpha > 0$ do not satisfy (2.10), that is, if
\[ \left( p > 1 \text{ and } \alpha \geq \frac{n + 2}{2p} \right) \quad \text{or} \quad \left( p = 1 \text{ and } \alpha > \frac{n + 2}{2p} \right) \] \hspace{1cm} (4.4)
then $J_\alpha f$ is generally not defined pointwise as an extended real valued function for $f \in X^p$. (However it can be defined for all $f$ in the subspace $L^p(\mathbb{R}^n \times \mathbb{R})$ of $X^p$ as a distribution on a certain subspace of the Schwarz space $S$ (see [21, Sec 9.2.5]).

Even though $J_\alpha f$ is generally not defined pointwise as and extended real valued function for $f \in X^p$ when $p$ and $\alpha$ satisfy (4.4), it is defined pointwise as a nonnegative extended real value
function for all nonnegative functions \( f \in X^p \) for all \( p \geq 1 \) and \( \alpha > 0 \) because then the integrand of \( J_\alpha f \) is a nonnegative function. Hence, since \( f \) is nonnegative in the problem (4.1)–(4.3), we see that the problem (4.1)–(4.3) makes sense for all \( p \geq 1 \) and \( \alpha > 0 \) when \( J_\alpha \) is defined in the pointwise sense, which is the sense in which we will define it in this section. However \( J_\alpha \), when restricted to the set \( X^+_p \) of all nonnegative functions \( f \in X^p \), is not one-to-one when \( p \) and \( \alpha \) satisfy (4.4). Thus our results in this section for the problem (4.1)–(4.3) when \( p \geq 1 \) and \( \alpha > 0 \) will yield corresponding results for the problem (3.1)–(3.3) only when \( p \) and \( \alpha \) satisfy (2.10).

In view of these remarks, we will consider in this section solutions \( f \in X^p \) of the following \( J_\alpha \) version of the fully fractional initial value problem (3.2), (3.3):

\[
0 \leq f \leq K(J_\alpha f)^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \quad n \geq 1
\]

\[
f = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0)
\]

where \( p \in [1, \infty) \) and \( K, \lambda, \alpha \in (0, \infty) \)

are constants, \( X^p \) is defined by (2.7), and \( J_\alpha \) is given by (2.4).

Under the equivalence of problems (3.1)–(3.3) and (4.1)–(4.3) discussed above, the following Theorems 4.1–4.6, when restricted to the case that \( p \) and \( \alpha \) satisfy (2.10) and \( K = 1 \), clearly imply Theorems 3.1–3.6 in Section 3. We will prove Theorems 4.1–4.6 in Section 8.

**Theorem 4.1.** Suppose \((\lambda, \alpha) \in A\) and \( f, p, \) and \( K \) satisfy (4.5)–(4.8). Then

\[
f = J_\alpha f = 0 \quad \text{almost everywhere in } \mathbb{R}^n \times \mathbb{R}.
\]

**Theorem 4.2.** Suppose \((\lambda, \alpha) \in B\) and \( f, p, \) and \( K \) satisfy (4.5)–(4.8). Then for all \( b > 0 \) we have

\[
\|f\|_{L^\infty(\mathbb{R}^n \times (0,b))} \leq K^{\frac{1}{1-\lambda}}(Mb^{\alpha})^{\frac{1}{1-\lambda}}
\]

and

\[
\|J_\alpha f\|_{L^\infty(\mathbb{R}^n \times (0,b))} \leq K^{\frac{1}{1-\lambda}}(Mb^{\alpha})^{\frac{1}{1-\lambda}}
\]

where

\[
M = M(\alpha, \lambda) = \frac{\Gamma\left(\frac{\alpha}{1-\lambda} + 1\right)}{\Gamma(\alpha + \frac{\alpha}{1-\lambda} + 1)}.
\]

**Theorem 4.3.** Suppose \( p \) and \( K \) satisfy (4.8), \((\lambda, \alpha) \in B\), \( T > 0 \), and \( 0 < N < M \) where \( M \) is given by (4.12). Then there exists a solution

\[
f \in L^p(\mathbb{R}^n \times \mathbb{R}) \cap C(\mathbb{R}^n \times \mathbb{R})
\]

of (4.6), (4.7) such that

\[
J_\alpha f \in C(\mathbb{R}^n \times \mathbb{R})
\]

\[
f(0, t) = K^{\frac{1}{1-\lambda}}(Nt^{\alpha})^{\frac{1}{1-\lambda}} \quad \text{for } 0 < t < T
\]

and

\[
J_\alpha f(0, t) \geq K^{\frac{1}{1-\lambda}}(Nt^{\alpha})^{\frac{1}{1-\lambda}} \quad \text{for } 0 < t < T.
\]
**Theorem 4.4.** Suppose $p$ and $K$ satisfy (4.8) and $(\lambda, \alpha) \in B$. Then there exists $N > 0$ and $f \in X^p$ satisfying (4.6), (4.7) such that

$$f(x, t) \geq K^{\frac{1}{1-\lambda}}(Nt^\alpha)^{\frac{1}{1-\lambda}} \quad \text{for} \quad |x|^2 < t$$

and

$$J_\alpha f(x, t) \geq K^{\frac{1}{1-\lambda}}(Nt^\alpha)^{\frac{1}{1-\lambda}} \quad \text{for} \quad |x|^2 < t.$$ (4.17)

**Theorem 4.5.** Suppose $p$ and $K$ satisfy (4.8),

$$(\lambda, \alpha) \in C \quad \text{and} \quad q \in (p, \infty).$$ (4.19)

Then there exists a solution

$$f \in L^p(\mathbb{R}^n \times \mathbb{R})$$ (4.20)

of (4.6), (4.7) and a sequence $\{t_j\} \subset (0, 1)$ such that

$$\lim_{j \to \infty} t_j = 0$$

and

$$\|f\|_{L^q(R_j)} = \infty \quad \text{for} \quad j = 1, 2, ..., \quad (4.21)$$

where

$$R_j = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |x| < \sqrt{t_j} \text{ and } t_j < t < 2t_j\}.$$ (4.22)

**Theorem 4.6.** Suppose $p$ and $K$ satisfy (4.8),

$$(\lambda, \alpha) \in C \quad \text{and} \quad \frac{n + 2}{2\alpha}(1 - \frac{1}{\lambda}) \leq q \leq \infty.$$ (4.23)

Then there exists a solution

$$f \in X^p$$ (4.24)

of (4.6), (4.7) and a sequence $\{t_j\} \subset (1, \infty)$ such that

$$\lim_{j \to \infty} t_j = \infty$$

and

$$\|f\|_{L^q(R_j)} = \infty \quad \text{for} \quad j = 1, 2, ..., \quad (4.25)$$

where $R_j$ is given in (4.22).

### 5 Preliminary results for fully fractional heat operators

In this section we provide some lemmas needed for the proofs of our results in Section 2 concerning the fully fractional heat operator (2.12).

The following lemma is needed for the proof of Theorem 2.2.

**Lemma 5.1.** Suppose $\alpha, \beta > 0$. Then

$$\Phi_{\alpha + \beta} = \Phi_\alpha \ast \Phi_\beta \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}$$

where $\Phi_\alpha$ is defined in (2.2).
Proof. Since
\[
\Phi_\alpha * \Phi_\beta(x, t) = \int_{-\infty}^{\infty} \int_{\xi \in \mathbb{R}^n} \Phi_\alpha(x - \xi, t - \tau) \Phi_\beta(\xi, \tau) \, d\xi \, d\tau
\]
\[
= \begin{cases} 
0 & \text{for } (x, t) \in \mathbb{R}^n \times (-\infty, 0] \\
\int_0^t \int_{\xi \in \mathbb{R}^n} \Phi_\alpha(x - \xi, t - \tau) \Phi_\beta(\xi, \tau) \, d\xi \, d\tau & \text{for } (x, t) \in \mathbb{R}^n \times (0, \infty),
\end{cases}
\]
we have (5.1) holds in \(\mathbb{R}^n \times (-\infty, 0]\).

Using the well-known facts that
\[
\hat{\Phi}_\alpha(\cdot, t)(y) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-|y|^2 t} \quad \text{for } t > 0 \text{ and } y \in \mathbb{R}^n \tag{5.3}
\]
and
\[
\int_0^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} \, d\tau = \frac{t^{\alpha+1-1}}{\Gamma(\alpha+1)} \quad \text{for } t, \alpha > 0, \tag{5.4}
\]
and assuming we can interchange the order of integration in the following calculation (we will justify this after the calculation) we obtain for \(t > 0\) and \(y \in \mathbb{R}^n\) that
\[
(\Phi_\alpha * \Phi_\beta)(\cdot, t)(y) = \int_{x \in \mathbb{R}^n} e^{ix \cdot y} \int_0^t \left( \int_{\xi \in \mathbb{R}^n} \Phi_\alpha(x - \xi, t - \tau) \Phi_\beta(\xi, \tau) \, d\xi \right) \, d\tau \, dx \tag{5.5}
\]
\[
= \int_0^t \left( \int_{x \in \mathbb{R}^n} e^{ix \cdot y} \left( \int_{\xi \in \mathbb{R}^n} \Phi_\alpha(x - \xi, t - \tau) \, dx \right) \Phi_\beta(\xi, \tau) \, d\xi \right) \, d\tau \tag{by the convolution theorem}
\]
\[
= e^{-\frac{|y|^2}{2} t} \int_0^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} \Phi_\beta(\xi, \tau) \, d\tau \quad \text{(by the convolution theorem)}
\]
\[
= e^{-\frac{|y|^2}{2} t} \frac{t^{\alpha+1-1}}{\Gamma(\alpha+1)} = \hat{\Phi}_{\alpha+\beta}(\cdot, t)(y). \tag{5.6}
\]

This calculation is justified by Fubini’s theorem and the fact that the integral (5.5) with \(e^{ix \cdot y}\) replaced with 1 is, by Fubini’s theorem for nonnegative functions and (5.4), equal to
\[
\int_0^t \int_{\xi \in \mathbb{R}^n} \left( \int_{x \in \mathbb{R}^n} \Phi_\alpha(x - \xi, t - \tau) \, dx \right) \Phi_\beta(\xi, \tau) \, d\xi \, d\tau
\]
\[
= \int_0^t \int_{\xi \in \mathbb{R}^n} \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} \Phi_\beta(\xi, \tau) \, d\xi \, d\tau
\]
\[
= \frac{t^{\alpha+1-1}}{\Gamma(\alpha+1)} \quad \text{for } t > 0 \text{ and } y \in \mathbb{R}^n.
\]

It follows now from (5.6) that (5.1) holds in \(\mathbb{R}^n \times (0, \infty)\). \(\square\)

The following lemma is needed for the proof of Lemma 5.3 which in turn is needed for the proof of Theorem 2.3.
Lemma 5.2. Suppose $f \in L^1(-\infty, 0)$ and $0 < \alpha \leq 1$. Then

$$g(t) := \int_{-\infty}^{t} (t-\tau)^{\alpha-1}|f(\tau)|d\tau < \infty \quad \text{for almost all } t \in (-\infty, 0).$$

Proof. The lemma is clearly true if $\alpha = 1$. Hence we can assume $0 < \alpha < 1$. Since

$$\int_{-\infty}^{0} (-t)^{-\alpha}g(t)\,dt = \int_{-\infty}^{0} (-t)^{-\alpha} \int_{-\infty}^{t} (t-\tau)^{\alpha-1}|f(\tau)|\,d\tau\,dt$$

$$= \int_{-\infty}^{0} |f(\tau)| \left( \int_{\tau}^{0} (-t)^{(1-\alpha)-1}(t-\tau)^{\alpha-1}\,dt \right)\,d\tau$$

$$= \Gamma(1-\alpha)\Gamma(\alpha) \int_{-\infty}^{0} |f(\tau)|\,d\tau < \infty,$$

where we have used (5.4), we see that $g(t) < \infty$ for almost all $t \in (-\infty, 0).$ \(\square\)

Lemma 5.3. Suppose $f \in L^1(\mathbb{R}^n \times (-\infty, 0))$, $\alpha \in (0, 1]$, and $y \in \mathbb{R}^n$. Then for almost all $t \in (-\infty, 0)$ we have

$$\mathcal{J}_\alpha f(\cdot, t)(y) = \int_{-\infty}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} e^{-|y|^2(t-\tau)} \hat{f}(\cdot, \tau)(y)\,d\tau.$$

Proof. By Fubini’s theorem for nonnegative functions and Lemma 5.2 we find for almost all $t \in (-\infty, 0)$ that

$$\int_{x \in \mathbb{R}^n} |e^{ix \cdot y}| \int_{-\infty}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \int_{\xi \in \mathbb{R}^n} \Phi_1(x-\xi, t-\tau)|f(\xi, \tau)|\,d\xi\,d\tau\,dx$$

$$= \int_{-\infty}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left( \int_{\xi \in \mathbb{R}^n} |f(\xi, \tau)|\,d\xi \right)\,d\tau < \infty.$$

Hence by Fubini’s theorem, the convolution theorem for Fourier transforms, and (5.3), we see for almost all $t \in (-\infty, 0)$ that

$$\mathcal{J}_\alpha f(\cdot, t)(y) = \int_{-\infty}^{t} \int_{x \in \mathbb{R}^n} e^{ix \cdot y} \int_{\xi \in \mathbb{R}^n} \Phi_\alpha(x-\xi, t-\tau)f(\xi, \tau)\,d\xi\,dx\,d\tau$$

$$= \int_{-\infty}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} e^{-|y|^2(t-\tau)} \hat{f}(\cdot, \tau)(y)\,d\tau.$$

\(\square\)

6 Fully fractional heat operator proofs

In this section we prove our fully fractional heat operator results which we stated in Section 2.

Proof of Theorem 2.1. Part (i) was proved by Sampson [22, Theorem 2.2]. We prove part (ii) in two steps.

Step 1. Suppose $f, g \in S$. Let $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ be momentarily fixed and define $\varphi \in S$ by

$$\varphi(y, s) = f(x + y, t + s).$$
Then
\[ \hat{\varphi}(y, s) = (2\pi)^{n+1} \varphi(-y, -s) = (2\pi)^{n+1} f(x - y, t - s) \]
and
\[ \hat{\varphi}(y, s) = e^{-ixy - its} \hat{f}(y, s). \]
Thus by part (i) with \( \varphi \) replaced with \( \hat{\varphi} \) we get
\[ (2\pi)^{n+1} J_\alpha f(x, t) = (2\pi)^{n+1} \int \int_{\mathbb{R}^n \times \mathbb{R}} \Phi_\alpha(y, s) f(x - y, t - s) dy ds \]
\[ = \int \int_{\mathbb{R}^n \times \mathbb{R}} \Phi_\alpha(y, s) \hat{\varphi}(y, s) dy ds \]
\[ = \int \int_{\mathbb{R}^n \times \mathbb{R}} (|y|^2 - is)^{-\alpha} \hat{\varphi}(y, s) dy ds \]
\[ = \int \int_{\mathbb{R}^n \times \mathbb{R}} (|y|^2 - is)^{-\alpha} \hat{f}(y, s) e^{-ixy - its} dy ds. \] (6.1)
Multiplying (6.1) by \( \hat{g}(x, t)/(2\pi)^{n+1} \), integrating the resulting equation with respect to \( (x, t) \), and interchanging the order of integration in the resulting integral on the RHS, which is allowed by Fubini's theorem and the fact that
\[ \| \hat{g} \|_{L^\infty(\mathbb{R}^n \times \mathbb{R})} \]
\[ \| g \|_{L^\infty(\mathbb{R}^n \times \mathbb{R})} \]
we get (2.5).

Step 2. Suppose \( f \in L^1(\mathbb{R}^n \times \mathbb{R}) \) and \( g \in S \). Then \( \hat{g} \in S \) and \( \hat{f} \in C(\mathbb{R}^n \times \mathbb{R}) \cap L^\infty(\mathbb{R}^n \times \mathbb{R}) \). Since \( S \) is dense in \( L^1(\mathbb{R}^n \times \mathbb{R}) \) there exists \( \{f_j\} \subset S \) such that \( f_j \to f \) in \( L^1(\mathbb{R}^n \times \mathbb{R}) \) and by Step 1
\[ \int \int_{\mathbb{R}^n \times \mathbb{R}} J_\alpha f_j(x, t) \hat{g}(x, t) dx dt = \int \int_{\mathbb{R}^n \times \mathbb{R}} (|y|^2 - is)^{-\alpha} \hat{f}_j(y, s) g(y, s) dy ds. \] (6.3)
Since
\[ \| \hat{f}_j - \hat{f} \|_{L^\infty(\mathbb{R}^n \times \mathbb{R})} \leq \| f_j - f \|_{L^1(\mathbb{R}^n \times \mathbb{R})} \to 0 \quad \text{as } j \to \infty \]
we have
\[ \left| \int \int_{\mathbb{R}^n \times \mathbb{R}} (\hat{f}_j(y, s) - \hat{f}(y, s))(|y|^2 - is)^{-\alpha} g(y, s) dy ds \right| \]
\[ \leq \| \hat{f}_j - \hat{f} \|_{L^\infty(\mathbb{R}^n \times \mathbb{R})} \int \int_{\mathbb{R}^n \times \mathbb{R}} |y|^2 - is)^{-\alpha} |g(y, s)| dy ds \]
\[ \to 0 \quad \text{as } j \to \infty \]
by (6.2). Thus the RHS of (6.3) tends to the RHS of (2.5) as \( j \to \infty \).

Also, defining \( h(x, t) = |\hat{g}(-x, -t)| \) we have
\[ \left| \int \int_{\mathbb{R}^n \times \mathbb{R}} J_\alpha (f_j - f)(x, t) \hat{g}(x, t) dx dt \right| \]
\[ \leq \int \int_{\mathbb{R}^n \times \mathbb{R}} \int \int_{\mathbb{R}^n \times \mathbb{R}} \Phi_\alpha(x - y, t - s)|(f_j - f)(y, s)| dy ds |\hat{g}(x, t)| dx dt \]
\[ = \int \int_{\mathbb{R}^n \times \mathbb{R}} |(f_j - f)(y, s)|(\Phi_\alpha * h)(-y, -s) dy ds \]
\[ \to 0 \quad \text{as } j \to \infty \]
because noting that $h \in L^1(\mathbb{R}^n \times \mathbb{R}) \cap L^\infty(\mathbb{R}^n \times \mathbb{R})$,

$$\|\Phi_\alpha \chi_{\mathbb{R}^n \times (0,1)}\|_{L^1(\mathbb{R}^n \times \mathbb{R})} = \int_0^1 \int_{x \in \mathbb{R}^n} \Phi_1(x, t) \, dx \, dt = \int_0^1 \frac{\Gamma(1)}{\Gamma(\alpha)} \, dt < \infty \quad \text{for } \alpha > 0,$$

(6.4)

and $\Phi_\alpha \chi_{\mathbb{R}^n \times (1,\infty)} \in L^\infty(\mathbb{R}^n \times \mathbb{R})$ for $\alpha < (n+2)/2$ we find that

$$\Phi_\alpha \ast h = \Phi_\alpha \chi_{\mathbb{R}^n \times (0,1)} \ast h + \Phi_\alpha \chi_{\mathbb{R}^n \times (1,\infty)} \ast h \in L^\infty(\mathbb{R}^n \times \mathbb{R})$$

by Young’s inequality. Thus the LHS of (6.3) tends to the LHS of (2.5) as $j \to \infty$. □

Proof of Theorem 2.2. Since

$$\bigcap_{T \in \mathbb{R}} L^p_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^T) = L^p_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})$$

and since $(J_\alpha f_T)|_{\mathbb{R}^n \times \mathbb{R}^T} = (J_\alpha f)|_{\mathbb{R}^n \times \mathbb{R}^T}$, where $f_T = f \chi_{\mathbb{R}^n \times \mathbb{R}^T}$ to prove (i), (ii) and (iii) it suffices to prove for all $T \in \mathbb{R}$ that

(i)' $J_\alpha f_T, J_\alpha |f_T| \in L^p_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^T)$

(ii)' $J_\beta J_\gamma f_T = J_\alpha f_T$ in $L^p_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^T)$ whenever $\beta > 0, \gamma > 0$, and $\beta + \gamma = \alpha$

and

(iii)' $HJ_\alpha f_T = J_{\alpha-1} f_T$ in $D'(\mathbb{R}^n \times \mathbb{R}^T)$ when $\alpha > 1$.

To do this, let $T \in \mathbb{R}$ be fixed. Since $f \in X^p \subset L^p(\mathbb{R}^n \times \mathbb{R}^T)$ we have

$$f_T \in L^p(\mathbb{R}^n \times \mathbb{R}).$$

(6.5)

Proof of (i)'. Since $|J_\alpha f_T| \leq J_\alpha |f_T|$, to prove (i)' it suffices to prove only that

$$J_\alpha |f_T| \in L^p_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^T).$$

(6.6)

By (2.3) we have

$$J_\alpha |f_T| = u_1 + u_2,$$

(6.7)

where

$$u_1 = (\Phi_\alpha \chi_{\mathbb{R}^n \times (0,1)}) \ast |f_T| \quad \text{and} \quad u_2 = (\Phi_\alpha \chi_{\mathbb{R}^n \times (1,\infty)}) \ast |f_T|.$$ 

It follows from (6.4), (6.5), and Young’s inequality that

$$u_1 \in L^p(\mathbb{R}^n \times \mathbb{R}).$$

Thus to complete the proof of (6.6) and hence of (i)' it suffices to show

$$u_2 \in L^\infty(\mathbb{R}^n \times \mathbb{R}).$$

(6.8)

To do this we consider two cases.
Case I. Suppose $1 < p < \frac{n+2}{2\alpha}$. Let $q$ be the conjugate Hölder exponent for $p$. Then

$$\frac{1}{q} = 1 - \frac{1}{p} < 1 - \frac{2\alpha}{n+2} = \frac{n+2 - 2\alpha}{n+2}$$

and thus making the change of variables $\sqrt{\frac{q}{4}} y = z$ we obtain

$$\|\Phi_{\alpha} \chi_{\mathbb{R}^n \times (1,\infty)}\|_{L^q(\mathbb{R}^n \times \mathbb{R})} = C(n,\alpha,q) \int_1^\infty \int_{y \in \mathbb{R}^n} s^{(\alpha-1-n/2)q} e^{-\frac{q}{4}|y|^2} dy ds$$

$$= C(n,\alpha,q) \int_1^\infty s^{(\alpha-1-n/2)q+n/2} \int_{z \in \mathbb{R}^n} e^{-|z|^2} dz ds < \infty.$$ 

Hence (6.8) follows from (6.5) and Young’s inequality.

Case II. Suppose $1 = p \leq \frac{n+2}{2\alpha}$. Then

$$\Phi_{\alpha} \chi_{\mathbb{R}^n \times (1,\infty)}(y,s) \leq C(n,\alpha) s^{\alpha-1-n/2} \chi_{\mathbb{R}^n \times (1,\infty)}(y,s) \leq C(n,\alpha) \text{ for } (y,s) \in \mathbb{R}^n \times \mathbb{R}.$$ 

Thus (6.8) follows from (6.5) and so the proof of (i)' is complete.

Proof of (ii)'. Using Fubini’s theorem for nonnegative functions and Lemma 5.1 we have

$$J_{\beta} (J_{\gamma}|f_T|)(x,t) = \int_{\mathbb{R}^n \times \mathbb{R}} \Phi_{\beta} (x - \xi, t - \tau) \int_{\mathbb{R}^n \times \mathbb{R}} \Phi_{\gamma} (\xi - \eta, \tau - \zeta) |f_T(\eta, \zeta)| d\eta d\zeta d\xi d\tau$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}} \Phi_{\beta+\gamma} (x - \eta, t - \zeta) |f_T(\eta, \zeta)| d\eta d\zeta$$

$$= (J_{\alpha}|f_T|)(x,t) < \infty \quad \text{a.e. in } \mathbb{R}^n \times \mathbb{R}$$

by part (i)'. Hence by Fubini’s theorem the above calculation can be repeated with $|f_T|$ replaced with $f_T$ which gives (ii)'.

Proof of (iii)'. By (i)' we have

$$J_{\alpha}|f_T|, \ J_{\alpha-1}|f_T| \in L^p_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}_T) \subset D'(\mathbb{R}^n \times \mathbb{R}_T).$$

(6.9)

Let $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_T)$. Then noting that

$$\int_{\mathbb{R}^n \times \mathbb{R}_T} \Phi_1 (x - \eta, t - \zeta) H^* \varphi(x,t) dx dt = \varphi(\eta, \zeta) \quad \text{for } (\eta, \zeta) \in \mathbb{R}^n \times \mathbb{R}_T$$

(6.10)

where $H^* = -\partial_t - \Delta$ and assuming we can interchange the order of integration in the following
calculation (we will justify this after the calculation) it follows from Lemma 5.1 that

\[
(H(J_{\alpha}f_T))(\varphi) = (J_{\alpha}f_T)(H^*\varphi)
\]

\[
\int \int_{\mathbb{R}^n \times \mathbb{R}_T} \left( \int \int_{\mathbb{R}^n \times \mathbb{R}_T} \Phi_\alpha(x - \xi, t - \tau) f_T(\xi, \tau) \, d\xi \, d\tau \right) H^*\varphi(x, t) \, dx \, dt
\]

\[
= \int \int_{\mathbb{R}^n \times \mathbb{R}_T} \left( \int \int_{\mathbb{R}^n \times \mathbb{R}_T} \Phi_1(x - \eta, t - \zeta) \Phi_{\alpha - 1}(\eta - \xi, \zeta - \tau) \, d\eta \, d\zeta \right)
\times f_T(\xi, \tau) \, d\xi \, d\tau \quad (6.12)
\]

\[
= \int \int_{\mathbb{R}^n \times \mathbb{R}_T} \left( \int \int_{\mathbb{R}^n \times \mathbb{R}_T} \Phi_{\alpha - 1}(\eta - \xi, \zeta - \tau) f_T(\xi, \tau) \, d\xi \, d\tau \right) \varphi(\eta, \zeta) \, d\eta \, d\zeta
\]

\[= (J_{\alpha - 1}f_T)(\varphi).\]

To justify this calculation, it suffices by Fubini’s theorem to show the integral (6.12), with \(f_T\) and \(H^*\varphi\) replaced with \(|f_T|\) and \(|H^*\varphi|\), is finite. However in the same way that (6.12) was obtained from (6.11), we see that this modified integral equals

\[
\int \int_{\mathbb{R}^n \times \mathbb{R}_T} (J_{\alpha}|f_T|)(x, t)|H^*\varphi|(x, t) \, dx \, dt < \infty
\]

by (6.9).

\[
\square
\]

Proof of Theorem 2.3. Clearly (ii) implies (i). We now prove (ii). Suppose (2.11). It follows from (2.4) that

\[
f|_{\mathbb{R}^n \times \mathbb{R}_T} = 0 \implies (J_{\alpha}f)|_{\mathbb{R}^n \times \mathbb{R}_T} = 0.
\]

Conversely suppose

\[ (J_{\alpha}f)|_{\mathbb{R}^n \times \mathbb{R}_T} = 0. \]

(6.13)

The complete the proof of (ii) it suffices to prove

\[ f|_{\mathbb{R}^n \times \mathbb{R}_T} = 0. \]

(6.14)

By Theorem 2.2(iii) and mathematical induction, we can, without loss of generality, assume for the proof (6.14) that

\[ 0 < \alpha \leq 1. \]

(6.15)

Moreover, by translating we can assume

\[ T = 0. \]

(6.16)

We divide the proof of (6.14) into two cases.

Case I. Suppose (2.10)_2 holds. Then

\[ 1 = p \leq \frac{n + 1}{2\alpha}. \]

(6.17)
Let
\[ F(y, t) = \hat{f}(\cdot, t)(y) \quad \text{for} \ (y, t) \in \mathbb{R}^n \times (-\infty, 0). \] (6.18)
By (2.11) and (6.17) we have
\[ f \in L^1(\mathbb{R}^n \times (-\infty, 0)) \] (6.19)
and thus
\[ f(\cdot, t) \in L^1(\mathbb{R}^n) \quad \text{for almost all} \ t \in (-\infty, 0) \]
which implies
\[ F(\cdot, t) \in C(\mathbb{R}^n) \quad \text{for almost all} \ t \in (-\infty, 0). \]
Also, by (6.19)
\[ \|F(y, \cdot)\|_{L^1(-\infty, 0)} = \int_{-\infty}^{0} \left| \int_{\mathbb{R}^n} e^{iy \cdot x} f(x, t) \, dx \right| \, dt \leq \|f\|_{L^1(\mathbb{R}^n \times (-\infty, 0))} < \infty \quad \text{for all} \ y \in \mathbb{R}^n. \] (6.20)

Case I(a). Suppose \( \alpha = 1 \). Then by (6.19), (6.13), and Lemma 5.3 we have for each \( y \in \mathbb{R}^n \) that
\[ \int_{-\infty}^{t} e^{\|y\|^2 \tau} F(y, \tau) \, d\tau = e^{\|y\|^2 t} \int_{-\infty}^{t} e^{-\|y\|^2 (t-\tau)} F(y, \tau) \, d\tau = 0 \]
for almost all \( t \in (-\infty, 0) \). Hence, by (6.20) and the measure theoretic fundamental theorem of calculus, we get \( F = 0 \) in \( L^1(\mathbb{R}^n \times (-\infty, 0)) \) which together with (6.18) implies (6.14).

Case I(b). Suppose \( 0 < \alpha < 1 \). To handle this case we hold \( y \in \mathbb{R}^n \setminus \{0\} \) fixed and define
\[ F_0(t) := F(y, t). \] (6.21)
Then by (6.20)
\[ F_0 \in L^1(-\infty, 0). \] (6.22)
From (6.19), (6.13), and Lemma 5.3 we have
\[ g(t) := \int_{-\infty}^{t} (t - \tau)^{\alpha-1} e^{\|y\|^2 \tau} F_0(\tau) d\tau = 0 \] (6.23)
for almost all \( t \in (-\infty, 0) \). On the other hand, assuming we can interchange the order of integration in the following calculation (we will justify this after the calculation), we find for \( b \in \mathbb{R} \) that
\[ \int_{-\infty}^{0} \left( \int_{t}^{0} (\zeta - t)^{-\alpha} \cos b \zeta d\zeta \right) g(t) \, dt \\
= \int_{-\infty}^{0} e^{\|y\|^2 \tau} F_0(\tau) \left( \int_{\tau}^{\zeta} \cos b \zeta \left( \int_{\tau}^{\zeta} (t - \tau)^{\alpha-1} (\zeta - t)^{-\alpha} d\tau \right) d\zeta \right) \, d\tau \\
= C(\alpha) \int_{-\infty}^{0} e^{\|y\|^2 \tau} F_0(\tau) \left( \int_{\tau}^{0} \cos b \zeta d\zeta \right) \, d\tau \] (6.24)
because making the change of variables \( t = \zeta - (\zeta - \tau)s \) we see that
\[ \int_{\tau}^{\zeta} (t - \tau)^{\alpha-1} (\zeta - t)^{-\alpha} \, dt = \int_{0}^{1} (1 - s)^{\alpha-1} s^{-\alpha} \, ds = C(\alpha). \]
The calculation (6.24) is justified by Fubini’s theorem and the fact that if we replace \( \cos b \zeta \) and \( g(t) \) with \( | \cos b \zeta | \) and respectively in the above calculation we get by Fubini’s theorem for nonnegative functions that
\[
\int_{-\infty}^{0} \int_{t}^{0} (\zeta - t)^{-\alpha} | \cos b \zeta | \, d\zeta \left( \int_{t}^{0} d\zeta \right) \leq C(\alpha) \int_{-\infty}^{0} e^{y|2\tau|} \left( \int_{\tau}^{0} | \cos b \zeta | \, d\zeta \right) d\tau < \infty
\]
by (6.22)

It follows now from (6.23), (6.24) and (6.21) that
\[
0 = \int_{-\infty}^{0} e^{y|2\tau|} F(y, \tau) \sin b \tau \, d\tau
\]
for all \( y \in \mathbb{R}^n \setminus \{0\} \) and all \( b \in \mathbb{R} \). Thus since the Fourier sine transform is one to one on \( L^1(\mathbb{R}^n) \) we have \( F(y, \cdot) = 0 \) in \( L^1(\mathbb{R}^n) \) for all \( y \in \mathbb{R}^n \setminus \{0\} \). Hence by Fubini’s theorem, \( F = 0 \) in \( L^1(\mathbb{R}^n \times (-\infty, 0)) \), which together with (6.18) and (6.16) implies (6.14).

**Case II.** Suppose (2.10) holds. Let \( f_T = f \chi_{\mathbb{R}^n \times \mathbb{R}^T} \) and \( u = J_\alpha f_T \). Then by (2.11) we have
\[
f_T \in L^p(\mathbb{R}^n \times \mathbb{R}),
\]
and by (2.4) and (6.13) we have
\[
u = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}.
\]
(6.25)
Let \( J_\varepsilon^{-\alpha} u \) be as defined in Theorem A.1. By (6.25) we have for \( l > \alpha \) that \( (\Delta^l_{y,\tau} u)(x, t) = 0 \) for \( (x, t) \in \mathbb{R}^n \times \mathbb{R}^T \) and \( (y, \tau) \in \mathbb{R}^n \times (0, \infty) \). Thus for \( \varepsilon > 0 \) we have
\[
J_\varepsilon^{-\alpha} u = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}.
\]
Hence (6.14) follows from Theorem A.1.

**7 Preliminary results for \( J_\alpha \) problems**

In this section we provide some lemmas needed for the proofs of our results in Section 4 dealing with solutions of the \( J_\alpha \) problem (4.5)–(4.8).

Let \( \Omega = \mathbb{R}^n \times (a, b) \) where \( n \geq 1 \) and \( a < b \). The following two lemmas give estimates for the convolution
\[
(V_{\alpha, \Omega}, f)(x, t) = \int_{\Omega} \Phi_\alpha(x - \xi, t - \tau) f(\xi, \tau) \, d\xi \, d\tau
\]
where \( \alpha > 0 \) and \( \Phi_\alpha \) is defined in (2.2).

**Remark 7.1.** Note that if \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is a nonnegative measurable function such that \( \| f \|_{L^\infty(\mathbb{R}^n \times \mathbb{R}_a)} = 0 \) then
\[
V_{\alpha, \Omega} f = J_\alpha f \quad \text{in} \quad \Omega := \mathbb{R}^n \times (a, b).
\]
Lemma 7.1. For $\alpha > 0$, $\Omega = \mathbb{R}^n \times (a,b)$ and $f \in L^\infty(\Omega)$ we have
\[
\|V_{\alpha,\Omega}f\|_{L^\infty(\Omega)} \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^\infty(\Omega)}.
\]

Proof. The lemma is obvious if $\|f\|_{L^\infty(\Omega)} = 0$. Hence we can assume $\|f\|_{L^\infty(\Omega)} > 0$. Then for $(x,t) \in \Omega$
\[
\frac{|(V_{\alpha,\Omega}f)(x,t)|}{\|f\|_{L^\infty(\Omega)}} \leq \int_a^t \left| \int_{\xi \in \mathbb{R}^n} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \Phi_1(x-\xi, t-\tau) d\xi \right| d\tau
\]
\[
= \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \bigg|_{\tau=a}^{\tau=t}
\]
\[
\leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}.
\]
\[\square\]

Lemma 7.2. Let $p, q \in [1, \infty]$, $\alpha$, and $\delta$ satisfy
\[
0 \leq \delta := 1 - \frac{1}{p} - \frac{1}{q} < \frac{2\alpha}{n+2} < 1. \tag{7.2}
\]

Then $V_{\alpha,\Omega}$ maps $L^p(\Omega)$ continuously into $L^q(\Omega)$ and for $f \in L^p(\Omega)$ we have
\[
\|V_{\alpha,\Omega}f\|_{L^q(\Omega)} \leq M \|f\|_{L^p(\Omega)}
\]
where
\[
M = C(b-a)^{\frac{2\alpha-(n+2)\delta}{2}} \text{ for some constant } C = C(n, \alpha, \delta).
\]

Proof. Define $r \in [1, \infty)$ by
\[
1 - \frac{1}{r} = \delta \tag{7.3}
\]
and define $P_\alpha, \tilde{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by
\[
P_\alpha(x,t) = \Phi_\alpha(x,t) \chi_{(0,b-a)}(t)
\]
and
\[
\tilde{f}(x,t) = \begin{cases} f(x,t) & \text{if } (x,t) \in \Omega \\ 0 & \text{elsewhere.} \end{cases}
\]

Since for $t \in (a,b)$ and $\tau \in (a,t)$ we have $t-\tau \in (0,b-a)$ we see for $(x,t) \in \Omega$ that
\[
V_{\alpha,\Omega}f(x,t) = \int_a^t \int_{\xi \in \mathbb{R}^n} P_\alpha(x-\xi, t-\tau) f(\xi, \tau) d\xi d\tau
\]
\[
= \int_{\Omega} P_\alpha(x-\xi, t-\tau) f(\xi, \tau) d\xi d\tau
\]
\[
= (P_\alpha * \tilde{f})(x,t) \tag{7.4}
\]
where $*$ is the convolution operation in $\mathbb{R}^n \times \mathbb{R}$.
Also since
\[
\int_{\mathbb{R}^n} e^{-r|x|^2/(4t)} \, dx = \left(\frac{4\pi t}{r}\right)^{n/2}
\]
we have by (7.2) and (7.3) that
\[
\|P_\alpha\|_{L^r(\mathbb{R}^n \times \mathbb{R})} = \frac{1}{\Gamma(\alpha)}(4\pi)^{n/2} \left(\int_0^{b-a} t^{r(\alpha-1-n/2)} \left(\int_{x \in \mathbb{R}^n} e^{-r|x|^2/(4t)} \, dx\right) \, dt\right)^{1/r}
= C(n, \alpha, r) \left(\int_0^{b-a} t^{r(\alpha-1-n/2)+\frac{n}{2}} \, dt\right)^{1/r}
= C(n, \alpha, r) \frac{2^{\alpha-(n+2)}(b-a)^{\frac{2\alpha}{2}}}{2^{\alpha}}.
\]
Thus by (7.4), (7.2), (7.3), and Young’s inequality we have
\[
\|V_{\lambda, \Omega} f\|_{L^q(\Omega)} = \|P_\alpha \ast \tilde{f}\|_{L^q(\Omega)} \leq \|P_\alpha \ast \tilde{f}\|_{L^q(\mathbb{R}^n \times \mathbb{R})}
\leq \|P_\alpha\|_{L^r(\mathbb{R}^n \times \mathbb{R})} \|\tilde{f}\|_{L^p(\mathbb{R}^n \times \mathbb{R})}
\leq C(b-a) \frac{2^{\alpha-(n+2)}(b-a)^{\frac{2\alpha}{2}}}{2} \|f\|_{L^p(\Omega)}.
\]

\[\square\]

Lemma 7.3. Suppose \(f, p,\) and \(K\) satisfy (4.5)–(4.8) and \((\lambda, \alpha) \in A \cup B\). Then
\[
f \in X^\infty.
\]

Proof. Let \(T > 0\) be fixed. Then \(f \in L^p(\mathbb{R}^n \times [0, T])\) and to complete the proof it suffices to show
\[
f \in L^\infty(\mathbb{R}^n \times (0, T)). \tag{7.5}
\]
We consider two cases.

Case I. Suppose \(0 < \alpha < \frac{n+2}{2p}\). Then
\[
0 < \lambda < \frac{n + 2}{n + 2 - 2\alpha p}
\]
and thus there exists \(\varepsilon = \varepsilon(n, \lambda, \alpha, p) > 0\) such that
\[
\varepsilon < 2\alpha p, \quad 2\varepsilon < n + 2 - 2\alpha p, \quad \text{and} \quad \lambda < \frac{n + 2}{n + 2 - 2\alpha p + 2\varepsilon}.
\]
Suppose
\[
f \in L^{p_0}(\mathbb{R}^n \times (0, T)) \quad \text{for some} \quad p_0 \in \left[p, \frac{n + 2}{2\alpha}\right]. \tag{7.6}
\]
Then letting
\[
q = \frac{(n + 2)p_0}{n + 2 - 2\alpha p_0 + \varepsilon}
\]
we have
\[
\frac{1}{p_0} - \frac{1}{q} = \frac{2\alpha}{n + 2} - \frac{\varepsilon}{(n + 2)p_0} \in \left(0, \frac{2\alpha}{n + 2}\right).
\]
Hence by (4.7), Remark 7.1, and Lemma 7.2 we see that
\[ J_\alpha f \in L^q(\mathbb{R}^n \times (0, T)). \]
Thus by (4.6) we find that
\[ 0 \leq f \leq K(J_\alpha f)\lambda \in L^{q/\lambda}(\mathbb{R}^n \times (0, T)). \quad (7.7) \]
Since
\[ \frac{q/\lambda}{p_0} = \frac{n + 2}{\lambda(n + 2 - 2\alpha p_0 + \varepsilon)} \geq \frac{n + 2 - 2\alpha p_0 + 2\varepsilon}{n + 2 - 2\alpha p_0 + \varepsilon} = C(n, \lambda, \alpha, p) > 1 \]
we see that starting with \( p_0 = p \) and iterating a finite number of times the process of going from (7.6) to (7.7) yields
\[ f \in L^{p_0}(\mathbb{R}^n \times (0, T)) \text{ for some } p_0 > \frac{n + 2}{2\alpha}. \]
Hence (7.5) follows from (4.6) and Lemma 7.2.

**Case II.** Suppose \( \alpha \geq \frac{n+2}{2p} \). Clearly there exists \( \hat{\alpha} \in (0, \frac{n+2}{2p}) \) such that \((\lambda, \hat{\alpha}) \in A \cup B\). Then for \((x, t), (\xi, \tau) \in \mathbb{R}^n \times (0, T)\) we have
\[ \frac{\Phi_\alpha(x - \xi, t - \tau)}{\Phi_{\hat{\alpha}}(x - \xi, t - \tau)} = \frac{(t - \tau)^{\alpha - \hat{\alpha}} \Gamma(\hat{\alpha})}{\Gamma(\alpha)} \leq T^{\alpha - \hat{\alpha}} \Gamma(\hat{\alpha}) / \Gamma(\alpha) = C(T, \alpha, \hat{\alpha}). \]
Thus for \((x, t) \in \mathbb{R}^n \times (0, T)\) we have
\[ J_\alpha f(x, t) \leq C(T, \alpha, \hat{\alpha}) J_{\hat{\alpha}} f(x, t) \]
and hence by (4.6) we see that
\[ 0 \leq f \leq KC(T, \alpha, \hat{\alpha})^{\lambda}(J_{\hat{\alpha}} f)^{\lambda} \text{ almost everywhere in } \mathbb{R}^n \times (0, T). \]
It follows therefore from Case I that \( f \) satisfies (7.5).

**Lemma 7.4.** Suppose \( x \in \mathbb{R}^n \) and \( t, \tau \in (0, \infty) \) satisfy
\[ |x|^2 < t \quad \text{and} \quad \frac{t}{4} < \tau < \frac{3t}{4}. \quad (7.8) \]
Then
\[ \int_{|\xi|^2 < \tau} \Phi_1(x - \xi, t - \tau) \, d\xi \geq C(n) > 0 \]
where \( \Phi_\alpha \) is defined by (2.2).
Proof. Making the change of variables $z = \frac{x - \xi}{\sqrt{4(t - \tau)}}$, letting $e_1 = (1, 0, \ldots, 0)$, and using (7.8) and (2.2) we find that

$$
\int_{|\xi|^2 < \tau} \Phi_1(x - \xi, t - \tau) \, d\xi = \frac{1}{\pi^{n/2}} \int_{|z| < \frac{\sqrt{\pi}}{\sqrt{4(t - \tau)}}} e^{-|z|^2} \, dz
$$

\[
\geq \frac{1}{\pi^{n/2}} \int_{|z| < \frac{\sqrt{\pi}}{\sqrt{4(t - \tau)}}} e^{-|z|^2} \, dz
\]

\[
\geq \frac{1}{\pi^{n/2}} \int_{|z| < \frac{\sqrt{\pi}}{2\sqrt{\pi}}} e^{-|z|^2} \, dz
\]

$$
= C(n) > 0
$$

where in this calculation we used the fact that the integral of $e^{-|z|^2}$ over a ball is decreased if the absolute value of the center of the ball is increased or the radius of the ball is decreased. \qed

Lemma 7.5. For $\tau < t \leq T$ and $|x| \leq \sqrt{T - t}$ we have

$$
\int_{|\xi| < \sqrt{T - \tau}} \Phi_1(x - \xi, t - \tau) \, d\xi \geq C
$$

where $C = C(n)$ is a positive constant.

Proof. Making the change of variables $z = \frac{x - \xi}{\sqrt{4(t - \tau)}}$ and letting $e_1 = (1, 0, \ldots, 0)$ we get

$$
\int_{|\xi| < \sqrt{T - \tau}} \Phi_1(x - \xi, t - \tau) \, d\xi = \frac{1}{(4\pi)^{n/2}} \int_{|z| < \sqrt{T - \tau}} e^{-|z|^2/4} \, dz
$$

\[
= \frac{1}{(4\pi)^{n/2}} \int_{|z| < \frac{\sqrt{\pi}}{\sqrt{T - \tau}}} e^{-|z|^2/4} \, dz
\]

\[
\geq \frac{1}{(4\pi)^{n/2}} \int_{|z| < \frac{\sqrt{\pi}}{\sqrt{T - \tau}}} e^{-|z|^2/4} \, dz
\]  \hspace{1cm} (7.9)

\[
\geq \frac{1}{(4\pi)^{n/2}} \int_{|z| < 1} e^{-|z|^2/4} \, dz
\]

\[
\geq \frac{1}{(4\pi)^{n/2}} \int_{|z| < 1} e^{-|z|^2/4} \, dz
\]

\[
\geq \frac{1}{(4\pi)^{n/2}} \int_{|z| < 1} e^{-|z|^2/4} \, dz
\]

(7.10)

(7.11)

where the last two inequalities need some explanation. Since $|x| \leq \sqrt{T - t} < \sqrt{T - \tau}$, the center of the ball of integration in (7.9) is closer to the origin than the center of the ball of integration in (7.10). Thus, since the integrand is a decreasing function of $|z|$, we obtain (7.10). Since $\sqrt{T - \tau} \geq \sqrt{t - \tau}$, the ball of integration in (7.10) contains the ball of integration in (7.11) and hence (7.11) holds. \qed

Lemma 7.6. Suppose $\alpha > 0$, $\gamma > 0$, $p \geq 1$, and

$$
f_0(x, t) = \left(\frac{1}{t}\right)^{\frac{n+2}{2p} - \gamma} \chi_{\Omega_0}(x, t) \quad \text{where} \quad \Omega_0 = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |x|^2 < t\}.
$$

Then $f_0 \in X^p$ and

$$
C_1 \left(\frac{1}{t}\right)^{\frac{n+2}{2p} - \gamma - \alpha} \leq J_\alpha f_0(x, t) \leq C_2 \left(\frac{1}{t}\right)^{\frac{n+2}{2p} - \gamma - \alpha} \quad \text{for} \quad (x, t) \in \Omega_0
$$

where $C_1$ and $C_2$ are positive constants depending only on $n, \alpha, \gamma$, and $p$.  

21
Proof. For \( T > 0 \) we have
\[
\|f_0\|_{L^p(\mathbb{R}^n \times \mathbb{R})} = \int_0^T \int_{|x|<\sqrt{t}} \left( \frac{1}{t} \right)^{n+2-\gamma p} dx \, dt
\]
\[
= C(n) \int_0^T t^{\gamma p-1} dt < \infty
\]
because \( \gamma p > 0 \). Hence \( f_0 \in X^p \).
Also for \( (x,t) \in \mathbb{R}^n \times (0, \infty) \) we have
\[
J_{\alpha}f_0(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n} \Phi_{\alpha}(x-\xi, t-\tau)f_0(\xi, \tau) \, d\xi \, d\tau
\]
\[
= \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} \left( \frac{1}{\tau} \right)^{\frac{n+2}{2p}-\gamma} \left( \int_{|\xi|^2<\tau} \Phi_1(x-\xi, t-\tau) \, d\xi \right) d\tau. \tag{7.12}
\]
Hence by Lemma 7.4 we see for \( (x,t) \in \Omega_0 \) that
\[
J_{\alpha}f_0(x,t) \geq C(n, \alpha) \int_{t/4}^{\frac{3t}{4}} (t-\tau)^{\alpha-1} \left( \frac{1}{\tau} \right)^{\frac{n+2}{2p}-\gamma} \, d\tau
\]
\[
= C(n, \alpha) t^{\alpha-\frac{n+2}{2p}+\gamma} \int_{1/4}^{3/4} (1-s)^{\alpha-1} \left( \frac{1}{s} \right)^{\frac{n+2}{2p}-\gamma} \, ds \quad \text{where} \quad \tau = ts
\]
Moreover for \( (x,t) \in \mathbb{R}^n \times (0, \infty) \) and \( 0 < \tau < t/2 \) we have
\[
\int_{|\xi|^2<\tau} \Phi_1(x-\xi, t-\tau) \, d\xi \leq \frac{1}{\pi^{n/2}} \int_{|z-\frac{x-\xi}{\sqrt{4(t-\tau)}}|<\frac{\sqrt{\tau}}{\sqrt{4(t-\tau)}}|} e^{-|z|^2} \, dz \quad \text{where} \quad z = \frac{x-\xi}{\sqrt{4(t-\tau)}}
\]
and for \( (x,t) \in \mathbb{R}^n \times (0, \infty) \) and \( t/2 < \tau < t \) we have
\[
\int_{|\xi|^2<\tau} \Phi_1(x-\xi, t-\tau) \, d\xi \leq \int_{\mathbb{R}^n} \Phi_1(x-\xi, t-\tau) \, d\xi = 1.
\]
Thus by (7.12) for \( (x,t) \in \mathbb{R}^n \times (0, \infty) \) we have
\[
J_{\alpha}f_0(x,t) \leq C(n, \alpha) \left[ \int_{0}^{t/2} (t-\tau)^{\alpha-1} \left( \frac{1}{\tau} \right)^{\frac{n+2}{2p}-\gamma} \left( \frac{\tau}{t-\tau} \right)^{n/2} \, d\tau + \int_{t/2}^{t} (t-\tau)^{\alpha-1} \left( \frac{1}{\tau} \right)^{\frac{n+2}{2p}-\gamma} \, d\tau \right]
\]
\[
= C(n, \alpha) t^{\alpha} \left[ \int_{0}^{1/2} (1-s)^{\alpha-1} \left( \frac{1}{s} \right)^{\frac{n+2}{2p}-\gamma} \left( \frac{s}{1-s} \right)^{n/2} \, ds + \int_{1/2}^{1} (1-s)^{\alpha-1} \left( \frac{1}{s} \right)^{\frac{n+2}{2p}-\gamma} \, ds \right]
\]
\[
= C(n, \alpha, \gamma, p) t^{\alpha-\frac{n+2}{2p}+\gamma}
\]
because \( \alpha \) and \( \gamma \) are positive.
Lemma 7.7. Suppose $\alpha > 0$, $\gamma \in \mathbb{R}$, $0 \leq t_0 < T$, $p \in [1, \infty)$, and

$$f(x, t) = \left(\frac{1}{T - t}\right)^{\frac{n+2}{2p} - \gamma} x_{\Omega}(x, t)$$

where

$$\Omega = \{(x, t) \in \mathbb{R}^n \times (t_0, T) : |x| < \sqrt{T - t}\}.$$

Then

$$J_\alpha f(x, t) \geq C\left(\frac{1}{T - t}\right)^{\frac{n+2}{2p} - \gamma - \alpha}$$

for $(x, t) \in \Omega^+ := \{(x, t) \in \Omega : \frac{T + t_0}{2} < t < T\}$ where $C = C(n, \alpha, \gamma, p) > 0$. Moreover,

$$f \in L^p(\mathbb{R}^n \times \mathbb{R}) \text{ if and only if } \gamma > 0$$

(7.13)

and in this case

$$\|f\|^p_{L^p(\mathbb{R}^n \times \mathbb{R})} = C(n) \int_0^{T-t_0} s^{\gamma p - 1} ds.$$  

(7.14)

Proof. Since

$$\|f\|^p_{L^p(\mathbb{R}^n \times \mathbb{R})} = \int_0^T \int_{|x| < \sqrt{T-t}} (T-t)^{\gamma p - \frac{n+2}{2p}} dx dt$$

$$= C(n) \int_0^T (T-t)^{\gamma p - 1} dt = C(n) \int_0^{T-t_0} s^{\gamma p - 1} ds$$

we see that (7.13) and (7.14) hold.

Let $r = \frac{n+2}{2p} - \gamma - \alpha$. Then for $(x, t) \in \Omega$ we have

$$J_\alpha f(x, t) = \int_{t_0}^{t} (T - \tau)^{-r - \alpha} \int_{|\xi| < \sqrt{T-\tau}} \Phi_\alpha(x - \xi, t - \tau) d\xi d\tau$$

$$= C \int_{t_0}^{t} (T - \tau)^{-r - \alpha} (t - \tau)^{\alpha - 1} \left(\int_{|\xi| < \sqrt{T-\tau}} \Phi_1(x - \xi, t - \tau) d\xi\right) d\tau$$

$$\geq C \int_{t_0}^{t} (T - \tau)^{-r - \alpha} (t - \tau)^{\alpha - 1} d\tau, \text{ by Lemma 7.5,}$$

$$= C(T - t)^{-r} g\left(\frac{t - t_0}{T - t}\right)$$

where $g(z) = \int_0^z (\zeta + 1)^{-r - \alpha} \zeta^{\alpha - 1} d\zeta$ and where we made the change of variables $t - \tau = (T - t)\zeta$. Thus

$$J_\alpha f(x, t) \geq C(T - t)^{-r} \text{ for } (x, t) \in \Omega^+$$

because $\frac{t - t_0}{T - t} > 1$ in $\Omega^+$.

8 Proofs of results for $J_\alpha$ problems

In this section we prove our results stated in Section 4 concerning pointwise bounds for nonnegative solutions $f$ of (4.5)–(4.8). As explained in Section 4, these results immediately imply Theorems 3.1–3.6 in Section 3.
Remark 8.1. The function \( g : \mathbb{R}^n \times \mathbb{R} \to [0, \infty) \) defined by
\[
g(x, t) = g(t) = \begin{cases} 
(Mt^\alpha)^{\frac{\lambda}{1-\lambda}} & \text{for } t > 0 \\
0 & \text{for } t \leq 0,
\end{cases}
\]
where \( \alpha > 0, 0 < \lambda < 1 \), and \( M = M(\alpha, \lambda) \) is defined in (4.12), satisfies
\[
g = (J_\alpha g)^\lambda \text{ in } \mathbb{R}^n \times \mathbb{R} \tag{8.1}
\]
which can be verified using (5.4). Even though \( g \notin X^p \) for all \( p \geq 1 \), it will be useful in our analysis of solutions of (4.6), (4.7) which are in \( X^p \) for some \( p \geq 1 \).

Remark 8.2. It will be convenient to scale (4.6) as follows. Suppose \( K, \lambda, \alpha, T \in (0, \infty) \), \( \lambda \neq 1 \), and \( f, \bar{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) are nonnegative measurable functions such that \( f = \bar{f} = 0 \) in \( \mathbb{R}^n \times (\mathbb{R} \times (\mathbb{R} \times (-\infty, 0)) \) and
\[
f(x, t) = K \frac{1}{T^{1-\alpha}} \frac{x}{T^{1-\alpha}} \bar{f}(\bar{x}, \bar{t})
\]
where
\[
x = T^{1/2} \bar{x} \quad \text{and} \quad t = T \bar{t}.
\]
Then \( f \) satisfies
\[
0 \leq f \leq K (J_\alpha f)^\lambda \text{ in } \mathbb{R}^n \times \mathbb{R}
\]
if and only if \( \bar{f} \) satisfies
\[
0 \leq \bar{f} \leq (J_\alpha \bar{f})^\lambda \text{ in } \mathbb{R}^n \times \mathbb{R}.
\]
Moreover
\[
\frac{f(x, t)}{K \frac{1}{T^{1-\alpha}} \frac{x}{T^{1-\alpha}}} = \frac{\bar{f}(\bar{x}, \bar{t})}{\bar{t}^{\frac{\alpha}{1-\lambda}}} \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, \infty)
\]
and
\[
\frac{J_\alpha f(x, t)}{K \frac{1}{T^{1-\alpha}} \frac{x}{T^{1-\alpha}}} = \frac{J_\alpha \bar{f}(\bar{x}, \bar{t})}{\bar{t}^{\frac{\alpha}{1-\lambda}}} \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, \infty).
\]

Proof of Theorem 4.1. Suppose for contradiction that (4.9) is false. Then there exists \( T > 0 \) such that
\[
\|f\|_{L^\infty(\mathbb{R}^n \times \mathbb{R})} > 0.
\]
Hence by (4.7) there exists \( t_0 \in [0, T) \) such that
\[
\|f\|_{L^\infty(\mathbb{R}^n \times \mathbb{R})} \begin{cases} 
0 & \text{for } t \leq t_0 \\
> 0 & \text{for } t > t_0.
\end{cases}
\]
Thus by Remark 7.1, we have for all \( b > t_0 \) that
\[
J_\alpha f = V_{\alpha, \Omega_b} f \quad \text{in } \Omega_b
\]
where \( \Omega_b = \mathbb{R}^n \times (t_0, b) \) and \( V_{\alpha, \Omega} \) is defined by (7.1). Also, by Lemma 7.3,
\[
\|f\|_{L^\infty(\Omega_b)} \leq \|f\|_{L^\infty(\Omega_T)} < \infty \quad \text{for } t_0 < b < T.
\]
It follows therefore from (4.6) and Lemma 7.1 that for \( t_0 < b < T \) we have
\[
0 < K^{-1} \leq \frac{\|V_{\alpha, \Omega_b} f\|_{L^\infty(\Omega_b)}^{\lambda}}{\|f\|_{L^\infty(\Omega_b)}} \leq \left( \frac{b - t_0}{T} \right)^{\alpha} \|f\|_{L^\infty(\Omega_b)}^{\lambda - 1} \to 0 \quad \text{as } b \to t_0^+
\]
because \( \lambda \geq 1 \). This contradiction proves Theorem 4.1. \( \square \)
Proof of Theorem 4.2. By Remark 8.2 with $T = 1$ we can assume $K = 1$. For $b > 0$ we have by Lemma 7.3 that

$$f \in L^\infty(\mathbb{R}^n \times \mathbb{R}_b)$$

and by (4.6), (4.7), Remark 7.1 with $a = 0$, and Lemma 7.1 that

$$\|f\|_{L^\infty(\Omega_b)} \leq \|J_\alpha f\|_{L^\infty(\Omega_b)} \leq \left(\frac{b^\alpha}{\Gamma(\alpha + 1)} \|f\|_{L^\infty(\Omega_b)}\right)^\lambda$$

where $\Omega_b = \mathbb{R}^n \times (0, b)$. Thus, since $0 < \lambda < 1$, we see that

$$\|f\|_{L^\infty(\Omega_b)} \leq \left(\frac{b^\alpha}{\Gamma(\alpha + 1)}\right)^\frac{1}{1-\lambda} \text{ for all } b > 0. \tag{8.2}$$

Define \(\{\gamma_j\} \subset (0, \infty)\) by \(\gamma_1 = 1\) and

$$\gamma_{j+1} = (\bar{M} \gamma_j)^\lambda, \quad j = 1, 2, ..., \quad \text{where } \bar{M} = \Gamma(\alpha + 1)M. \tag{8.3}$$

Then, since $0 < \lambda < 1$, we see that

$$\gamma_j \to \bar{M}^\frac{1}{1-\lambda} \text{ as } j \to \infty. \tag{8.4}$$

Suppose for some positive integer $j$ that

$$\|f\|_{L^\infty(\Omega_b)} \leq \gamma_j \left(\frac{b^\alpha}{\Gamma(\alpha + 1)}\right)^\frac{1}{1-\lambda} \text{ for all } b > 0. \tag{8.5}$$

Then for $b > 0$ and $(x, t) \in \Omega_b$ we find from (4.6) and (5.4) that

$$f(x, t) \leq (J_\alpha f(x, t))^\lambda$$

$$\leq \left(\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_{\xi \in \mathbb{R}^n} \Phi_1(x - \xi, t-\tau) d\xi\right) \|f\|_{L^\infty(\Omega_\tau)} d\tau\right)^\lambda$$

$$\leq \left(\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \gamma_j \left(\frac{\tau^\alpha}{\Gamma(\alpha + 1)}\right)^\frac{1}{1-\lambda} d\tau\right)^\lambda$$

$$= \left(\gamma_j \frac{1}{\Gamma(\alpha) \Gamma(\alpha + 1)^{\frac{\alpha}{1-\lambda}}} \int_0^t (t-\tau)^{\alpha-1} \tau^{\frac{\alpha}{1-\lambda}} d\tau\right)^\lambda$$

$$= \left(\gamma_j \frac{\Gamma(\alpha) \Gamma(\frac{\alpha}{1-\lambda} + 1) t^{\alpha + \frac{\alpha}{1-\lambda}}}{\Gamma(\alpha) \Gamma(\alpha + 1)^{\frac{\alpha}{1-\lambda}} \Gamma(\alpha + \frac{\alpha}{1-\lambda} + 1)}\right)^\lambda$$

$$= \left(\gamma_j \frac{\bar{M} t^{\frac{\alpha}{1-\lambda}}}{\Gamma(\alpha + 1)^{\frac{\alpha}{1-\lambda}}}\right)^\lambda = \left(\gamma_j \frac{\bar{M} t^{\frac{\alpha}{1-\lambda}}}{\Gamma(\alpha + 1)^{\frac{1}{1-\lambda}}}\right)^\lambda \tag{8.6}$$

$$= \gamma_{j+1} \left(\frac{t^\alpha}{\Gamma(\alpha + 1)}\right)^\frac{1}{1-\lambda}.$$
Thus \[ \|f\|_{L^\infty(\Omega_b)} \leq \gamma_{j+1} \left( \frac{b\alpha}{\Gamma(\alpha + 1)} \right)^{\frac{1}{1-\lambda}} \] for all \( b > 0 \).

Hence (4.10) follows inductively from (8.2)–(8.5).

Finally, repeating the calculation (8.6) with \( \gamma_j = \gamma_{j+1} = \bar{M}^{\frac{1}{1-\lambda}} \) we get

\[ (J_{\alpha}f(x,t))^\lambda \leq \bar{M}^{\frac{1}{1-\lambda}} \left( \frac{t\alpha}{\Gamma(\alpha + 1)} \right)^{\frac{1}{1-\lambda}} \] for \((x,t) \in \Omega_b\)

which proves (4.11).

**Proof of Theorem 4.3.** By Remark 8.2 we can assume \( K = T = 1 \). For \((x,t) \in \mathbb{R}^n \times \mathbb{R} \) and \( \delta \in (0,1) \) let

\[ g_\delta(x,t) = g_\delta(t) = \psi_\delta(t)g(t) \quad (8.7) \]

where \( g \) is as in Remark 8.1 and \( \psi_\delta \in C^\infty(\mathbb{R} \to [0,1]) \) satisfies

\[ \psi_\delta(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ 0 & \text{if } t \geq 1 + \delta. \end{cases} \]

Then for \( 1 \leq t \leq 1 + \delta \)

\[ J_{\alpha}g(t) - J_{\alpha}g_\delta(t) = \int_1^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g(\tau)(1 - \psi_\delta(\tau)) \, d\tau \]

\[ \leq \int_1^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g(\tau) \, d\tau \leq g(1+\delta) \int_1^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \, d\tau \]

\[ = g(1+\delta) \frac{(t-1)^\alpha}{\Gamma(\alpha + 1)} \leq g(2) \frac{\delta^\alpha}{\Gamma(\alpha + 1)} \]

and thus by (8.1) we have for \( 1 \leq t \leq 1 + \delta \) that

\[ \frac{J_{\alpha}g_\delta(t)}{J_{\alpha}g(t)} = \frac{J_{\alpha}g(t) - (J_{\alpha}g(t) - J_{\alpha}g_\delta(t))}{g(t)^{1/\lambda}} \]

\[ \geq 1 - \frac{g(2) \delta^\alpha}{\Gamma(\alpha + 1) g(1)^{1/\lambda}} \]

\[ = 1 - C(\alpha, \lambda) \delta^\alpha \geq \sqrt{N} \]

provided we choose \( \delta = \delta(\alpha, \lambda, N) \in (0,1) \) sufficiently small. Hence for \( 1 \leq t \leq 1 + \delta \) we see from (8.1) that

\[ g_\delta(t) \leq g(t) = (J_{\alpha}g(t))^\lambda \leq \left( \frac{M}{N} \right)^{\lambda/2} (J_{\alpha}g_\delta(t))^\lambda \quad (8.8) \]

which by (8.7) and (8.1) holds for all other \( t \) as well.

Next let \( \varphi(x) = e^{-\psi(x)} \) where \( \psi(x) = \sqrt{1 + |x|^2} - 1 \). Then for \( \varepsilon \in (0,1) \), \( \gamma > 1 \), and \( |\xi - x| < \gamma \sqrt{2} \) we have

\[ \frac{\varphi(\varepsilon \xi)}{\varphi(\varepsilon x)} = e^{-(\psi(\varepsilon x) - \psi(\varepsilon x))} \geq e^{-\varepsilon |\xi - x|} \geq e^{-\varepsilon \gamma \sqrt{2}}. \]
Thus defining \( f_\varepsilon : \mathbb{R}^n \times \mathbb{R} \to [0, \infty) \) by

\[
f_\varepsilon(x, t) = \varphi(\varepsilon x) \left( \frac{N}{M} \right)^{\frac{1}{\alpha}} g_\delta(t)
\]
we find for \( |\xi - x| < \gamma \sqrt{2} \) and \( \tau \in \mathbb{R} \) that

\[
f_\varepsilon(\xi, \tau) \geq \varphi(\varepsilon x) e^{-\varepsilon \gamma \sqrt{2}} \left( \frac{N}{M} \right)^{\frac{1}{\alpha}} g_\delta(\tau).
\]

Thus for \( (x, t) \in \mathbb{R}^n \times (0, 2) \) we have

\[
J_\alpha f_\varepsilon(x, t) \geq \varphi(\varepsilon x) e^{-\varepsilon \gamma \sqrt{2}} \left( \frac{N}{M} \right)^{\frac{1}{\alpha}} \int_0^t \frac{(t-\tau)^{n-1}}{\Gamma(\alpha)} g_\delta(\tau) \int_{|\xi-x|<\gamma \sqrt{2}} \Phi_1(x - \xi, t - \tau) d\xi d\tau. \tag{8.9}
\]

But for \( x, \xi \in \mathbb{R}^n \) and \( 0 < \tau < t < 2 \) we find making the change of variables \( z = \frac{x - \xi}{\sqrt{4(t-\tau)}} \) that

\[
\int_{|\xi-x|<\gamma \sqrt{2}} \Phi_1(x - \xi, t - \tau) d\xi \geq \int_{|\xi-x|<\gamma \sqrt{1-t/\tau}} \frac{1}{4\pi(t-\tau)^{n/2}} e^{-\frac{|x-\xi|^2}{4(t-\tau)}} d\xi = \frac{1}{\pi^{n/2}} \int_{|z|<\gamma/2} e^{-|z|^2} dz =: I(\gamma) \to 1
\]
as \( \gamma \to \infty \). Thus by (8.9) and (8.8) we have for \( (x, t) \in \mathbb{R}^n \times (0, 1 + \delta) \) that

\[
\left( \frac{J_\alpha f_\varepsilon(x, t)}{f_\varepsilon(x, t)} \right)^\lambda \geq \varphi(\varepsilon x)^\lambda e^{-\varepsilon \gamma \lambda \sqrt{2}} \left( \frac{N}{M} \right)^{\frac{1}{\alpha}} I(\gamma)^\lambda \left( J_\alpha g_\delta(t) \right)^\lambda \geq \left( \frac{M}{N} \right)^{\lambda/2} I(\gamma)^\lambda e^{-\varepsilon \gamma \lambda \sqrt{2}}. \tag{8.10}
\]

So first choosing \( \gamma \) so large that \( \left( \frac{M}{N} \right)^{\lambda/2} I(\gamma)^\lambda > 1 \) and then choosing \( \varepsilon > 0 \) so small that (8.10) is greater that one we see that \( f := f_\varepsilon \) satisfies (4.6) in \( \mathbb{R}^n \times (0, 1 + \delta) \). Thus, since \( g_\delta(t) \) and hence \( f(x, t) \) is identically zero in \( \mathbb{R}^n \times ((-\infty, 0] \cup [1 + \delta, \infty)) \) see that \( f \) satisfies (4.6), (4.7).

From the exponential decay of \( \varphi(x) \) as \( |x| \to \infty \), we see that \( f \) satisfies (4.13). Also since \( f \) is uniformly continuous and bounded on \( \mathbb{R}^n \times \mathbb{R} \) and

\[
\int_a^b \int_{\mathbb{R}^n} \Phi_\alpha(x, t) \, dx \, dt = \frac{1}{\Gamma(\alpha + 1)} (b^\alpha - a^\alpha) \quad \text{for } a < b,
\]
we easily check that (4.14) holds.

Finally, since

\[
f(0,t) = \left( \frac{N}{M} \right)^{\frac{1}{\alpha}} g(t) \quad \text{for } 0 \leq t \leq 1
\]
we find that (4.15) holds and thus (4.16) follows from (4.6).

**Proof of Theorem 4.4.** By Remark 8.2 with \( T = 1 \) we can assume \( K = 1 \). Define \( \bar{f} : \mathbb{R}^n \times \mathbb{R} \to [0, \infty) \) by

\[
\bar{f}(x, t) = g(t) \chi_{\{|z|<\gamma \}}(x, t) \tag{8.11}
\]
where \( g \) is defined in Remark 8.1. Then for \((x,t) \in \mathbb{R}^n \times (0, \infty)\) we have

\[
J_\alpha \bar{f}(x,t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left( \int_{|\xi|^2 < \tau} \Phi_1(x - \xi, t - \tau) \, d\xi \right) g(\tau) \, d\tau.
\]

Thus by Lemma 7.4 we see for \(|x|^2 < t\) that

\[
J_\alpha \bar{f}(x,t) \geq C(n, \alpha, \lambda) \left( \frac{t}{4} \right)^{\alpha \lambda} \chi_{\Omega_0}(x,t) \quad (8.12)
\]

which also holds in \((\mathbb{R}^n \times \mathbb{R}) \setminus \{|x|^2 \leq t\}\) because \(\bar{f} = 0\) there. Thus letting \( f = L\bar{f} \) where

\[
L = C \frac{\lambda}{r}
\]

where \( C = C(n, \alpha, \lambda) \) is as in (8.12) we find that \( f \) satisfies (4.5)–(4.7).

It follows from (8.11) and the definitions of \( g \) and \( f \) that there exists \( N > 0 \) such that (4.17) holds. Thus, since \( f \) solves (4.6) we obtain (4.18).

**Proof of Theorem 4.5.** Since \( |R_j| < \infty \) we can assume \( q < \infty \). Thus by (4.19) there exists \( \varepsilon = \varepsilon(n, \lambda, \alpha, p, q) \in (0, 1) \) such that

\[
r := \frac{n+2}{2p} - \alpha - 2\varepsilon > 0, \quad \frac{(r + \alpha + 2\varepsilon)p}{r + \alpha} < q < \infty
\]

and

\[
\lambda > \frac{n + 2 - 4\varepsilon p}{n + 2 - 4\alpha p - 4\varepsilon p} = \frac{r + \alpha}{r}.
\]

Define \( f_0 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) by

\[
f_0(x,t) = \left( \frac{1}{t} \right)^{r + \alpha + \varepsilon = \frac{n+2}{2p} - \varepsilon} \chi_{\Omega_0}(x,t) \quad (8.15)
\]

where

\[
\Omega_0 = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x|^2 < t < 1\}.
\]

Then by Lemma 7.6 we have

\[
f_0 \in L^p(\mathbb{R}^n \times \mathbb{R}) \quad (8.16)
\]

and

\[
J_\alpha f_0(x,t) \geq C \left( \frac{1}{t} \right)^{r + \varepsilon} \quad \text{for } (x,t) \in \Omega_0 \quad (8.17)
\]

where, throughout this entire proof, \( C = C(n, \lambda, \alpha, p, q) \) is a positive constant whose value may change from line to line.

Let \( \{T_j\} \subset (0, 1/2) \) be a sequence such that

\[
T_{j+1} < T_j/4 \quad j = 1, 2, ... \]
and define \( t_j \in (0, T_j) \) by
\[
(T_j - t_j) = t_j^{(r+\varepsilon)/r}.
\] (8.18)
Then
\[
1 > \frac{T_j - t_j}{t_j} = t_j^{\varepsilon/r} \to 0 \quad \text{as} \quad j \to \infty.
\]
Thus \( 0 < t_j < T_j < 2t_j < 1 \) and therefore
\[
\Omega_j := \{ (y, s) \in \mathbb{R}^n \times \mathbb{R} : |y| < \sqrt{T_j - s} \text{ and } t_j < s < T_j \} \subset R_j \subset \Omega_0.
\] (8.19)
Defining \( f_j : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) by
\[
f_j(x, t) = (T_j - t)^{-r-\alpha} \chi_{\Omega_j}(x, t)
\]
and noting from (8.13) that
\[
r + \alpha = \frac{n + 2}{2p} - 2\varepsilon
\]
and
\[
r + \alpha > \frac{p}{q}(r + \alpha + 2\varepsilon) = \frac{p(n + 2)}{2pq} = \frac{n + 2}{2q},
\]
we obtain from Lemma 7.7 that
\[
\|f_j\|_{L^p(\mathbb{R}^n \times \mathbb{R})} = C(n) \int_0^{T_j - t_j} s^{2\varepsilon - 1} ds \to 0 \quad \text{as} \quad j \to \infty,
\] (8.20)
\[
\|f_j\|_{L^q(\Omega_j)} = \|f_j\|_{L^q(\mathbb{R}^n \times \mathbb{R})} = \infty \quad \text{for} \quad j = 1, 2, \ldots,
\] (8.21)
and
\[
J_{\alpha} f_j(x, t) \geq C(T_j - t)^{-r} \quad \text{for} \quad (x, t) \in \Omega_j^+
\] (8.22)
where
\[
\Omega_j^+ = \{ (x, t) \in \Omega_j : t - t_j > T_j - t \}.
\]
It follows from (8.15), (8.17), and (8.14) that for \((x, t) \in \Omega_0\) we have
\[
\frac{f_0(x, t)}{(J_{\alpha} f_0(x, t))^\lambda} \leq C t^{(r+\varepsilon)\lambda - (r+\alpha+\varepsilon)} \leq C t^{(r+\varepsilon)(r+\alpha)/r - (r+\alpha+\varepsilon)} = C t^{\varepsilon\alpha/r}.
\]
Thus
\[
\sup_{\Omega_0} \frac{f_0}{(J_{\alpha} f_0)^\lambda} \leq C
\] (8.23)
and by (8.19)
\[
\sup_{\Omega_j} \frac{f_0}{(J_{\alpha} f_0)^\lambda} \leq C T_j^{\varepsilon\alpha/r} < 1
\] (8.24)
by taking a subsequence.
By (8.22) and (8.14)
\[
\sup_{\Omega_j^+} \frac{f_j}{(J_{\alpha} f_j)^\lambda} \leq C \sup_{(x, t) \in \Omega_j^+} (T_j - t)^{r\lambda - (r+\alpha)} \leq C (T_j - t_j)^{r\lambda - (r+\alpha)} < 1
\] (8.25)
by taking a subsequence.

It follows from (8.15), (8.19), and (8.18) that

\[
\sup_{\Omega_j} f_0 \equiv \sup_{(x,t) \in \Omega_j} \frac{(T_j - t)^{r+\alpha}}{t^{r+\alpha + \varepsilon}} \leq \frac{(T_j - t_j)^{r+\alpha}}{t_j^{r+\alpha + \varepsilon}}
\]

\[
= \frac{t_j^{(r+\alpha)(r+\varepsilon)/r}}{t_j^{r+\alpha + \varepsilon}} = t_j^{\varepsilon/r} < 1
\]

(8.26)

and letting \(\Omega_j^- = \Omega_j \setminus \Omega_j^+\) we see from (8.17), (8.18), and (8.14) that

\[
\sup_{\Omega_j^-} f_j \leq C \sup_{(x,t) \in \Omega_j^-} \frac{t^{\lambda(r+\varepsilon)}}{(T_j - t)^{r+\alpha}} \leq C \frac{T_j^{\lambda(r+\varepsilon)}}{(T_j - t_j)^{r+\alpha}}
\]

\[
\leq C \left( \frac{2t_j}{t_j^{r+\alpha}} \right)^{\lambda(r+\varepsilon)} = C t_j^{(r+\varepsilon)(\lambda - \frac{r+\alpha}{r+\varepsilon})} < \frac{1}{2}
\]

(8.27)

by taking a subsequence.

Taking an appropriate subsequence of \(f_j\) and letting

\[ f = f_0 + \sum_{j=1}^{\infty} f_j \]

we find from (8.16) and (8.20) that \(f\) satisfies (4.20).

In \(\Omega_j^+\) we have by (8.24) and (8.25) that

\[ f = f_0 + f_j \leq (J_\alpha f_0)^\lambda + (J_\alpha f_j)^\lambda \leq (J_\alpha (f_0 + f_j))^\lambda \leq (J_\alpha f)^\lambda. \]

In \(\Omega_j^-\) we have by (8.26) and (8.27) that

\[ f = f_0 + f_j \leq 2f_j \leq (J_\alpha f_0)^\lambda \leq (J_\alpha f)^\lambda. \]

In \(\Omega_0 \setminus \bigcup_{j=1}^{\infty} \Omega_j\) we have by (8.23) that

\[ f = f_0 \leq C(J_\alpha f_0)^\lambda \leq C(J_\alpha f)^\lambda. \]

In \((\mathbb{R}^n \times \mathbb{R}) \setminus \Omega_0\), \(f = 0 \leq (J_\alpha f)^\lambda\). Thus, after scaling \(f\), we see that \(f\) is a solution of (4.6), (4.7). Also (4.21) holds by (8.21).

\[ \square \]

**Proof of Theorem 4.6.** By (4.23)\(_1\), there exists a unique number \(\gamma \in (0, \frac{n+2}{2p} - \alpha)\) such that

\[
\lambda = \frac{n+2}{2p} - \gamma - \frac{\alpha - \gamma}{n+2 - \alpha - \gamma}.
\]

(8.28)

Let \(f_0\) and \(\Omega_0\) be as in Lemma 7.6. Then by (8.28) and Lemma 7.6 we have

\[ f_0 \in X^p \]

(8.29)
and
\[ f_0 \leq C(J_\alpha f_0)^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R} \quad (8.30) \]
where in this proof \( C = C(n, \lambda, \alpha, p) \) is a positive constant whose value may change from line to line. Let \( \{T_j\}, \{t_j\} \subset (2, \infty) \) satisfy
\[ T_{j+1} \geq 4T_j \quad \text{and} \quad T_j = 2t_j \]
and define \( f_j : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) by
\[ f_j(x,t) = \left( \frac{1}{T_j - t} \right)^{\frac{n+2}{2p} - \gamma} \chi_{\Omega_j}(x,t) \quad (8.31) \]
where
\[ \Omega_j := \{(x,t) \in \mathbb{R}^n \times (T_j/2, T_j) : |x| < \sqrt{T_j - t}\} . \]
Then
\[ \Omega_j \subset R_j \subset \Omega_0, \quad \Omega_j \cap \Omega_k = \emptyset \quad \text{for } j \neq k, \quad (8.32) \]
\[ \inf \{t : (x,t) \in \Omega_j\} = T_j/2 \to \infty \quad \text{as } j \to \infty, \quad (8.33) \]
and by (8.31), (8.28), and Lemma 7.7 we have
\[ f_j \in L^p(\mathbb{R}^n \times \mathbb{R}) \quad (8.34) \]
and
\[ f_j \leq C(J_\alpha f_j)^\lambda \quad \text{in } \Omega_j^+ \quad (8.35) \]
where
\[ \Omega_j^+ = \{(x,t) \in \Omega_j : 3T_j/4 < t < T_j\} . \]
It follows therefore from (8.30) that
\[ f_0 + f_j \leq C((J_\alpha f_0)^\lambda + (J_\alpha f_j)^\lambda) \leq C(J_\alpha(f_0 + f_j))^\lambda \quad \text{in } \Omega_j^+. \quad (8.36) \]
In \( \Omega_j^- := \Omega_j \setminus \Omega_j^+ \) we have
\[ \frac{f_j}{f_0} = \left( \frac{t}{(T_j - t)} \right)^{\frac{n+2}{2p} - \gamma} \leq \left( \frac{3T_j}{4T_j} \right)^{\frac{n+2}{2p} - \gamma} = 3^{\frac{n+2}{2p} - \gamma} \]
and thus we obtain from (8.30) that
\[ f_0 + f_j \leq C(J_\alpha f_0)^\lambda \leq C(J_\alpha(f_0 + f_j))^\lambda \quad \text{in } \Omega_j^- . \quad (8.37) \]
Let \( f = f_0 + \sum_{j=1}^{\infty} f_j \). Then clearly \( f \) satisfies (4.7) and by (8.29), (8.34), and (8.33) we see that \( f \) satisfies (4.24).
In \( \Omega_j \) we have by (8.32)2, (8.35) and (8.36) that
\[ f = f_0 + f_j \leq C(J_\alpha(f_0 + f_j))^\lambda \leq C(J_\alpha f)^\lambda \]
and in \( (\mathbb{R}^n \times \mathbb{R}) \setminus \bigcup_{j=1}^{\infty} \Omega_j \) we have by (8.30) that
\[ f = f_0 \leq C(J_\alpha f_0)^\lambda \leq C(J_\alpha f)^\lambda . \]

31
Thus after scaling \( f \), we find that \( f \) satisfies (4.6).

Since \( |R_j| < \infty \), we can for the proof of (4.25) assume instead of (4.23) that

\[
q = \frac{n + 2}{2\alpha}(1 - \frac{1}{\lambda})
\]

and hence by (8.28) we get

\[
\frac{n + 2}{2p} - \gamma = \frac{\alpha}{1 - \frac{1}{\lambda}} = \frac{n + 2}{2q}.
\]

Consequently from (8.32), (8.31), and Lemma 7.7 we find that

\[
\|f\|_{L^q(R_j)} \geq \|f_j\|_{L^q(\Omega_j)} = \infty \quad \text{for } j = 1, 2, ...
\]

which proves (4.25) \( \square \)

A Appendix

For the proof of Theorem 2.3(ii) we will need the following result due to Nogin and Rubin \[16\] concerning the inversion of the operator \( J_\alpha \) in the framework of the spaces \( L^p(\mathbb{R}^n \times \mathbb{R}) \). See also [21, Theorem 9.24].

**Theorem A.1.** Suppose \( 0 < \alpha < \frac{n+2}{2p} \), \( 1 < p < \infty \), and \( u = J_\alpha f \) with \( f \in L^p(\mathbb{R}^n \times \mathbb{R}) \). Then

\[
\lim_{\varepsilon \to 0^+} J^{-\alpha}_\varepsilon u = f \quad \text{in } L^p(\mathbb{R}^n \times \mathbb{R})
\]

where

\[
J^{-\alpha}_\varepsilon u(x, t) = C(n, \alpha, l) \int_{\mathbb{R}^n \times (\varepsilon, \infty)} \frac{(\Delta_{y,\tau}^l u)(x, t)}{t^{1+\alpha}} e^{-\frac{|y|^2}{4\tau}} dy d\tau \quad (A.1)
\]

and

\[
(\Delta_{y,\tau}^l u)(x, t) = \sum_{k=0}^l (-1)^k \binom{l}{k} u(x - \sqrt{k\tau}y, t - k\tau), \quad l > \alpha. \quad (A.2)
\]

References


