CHAPTER 9 – VOTING

Warning! Do not use previous semester's notes (prior edition) for Borda Count or Condorcet's Method

Majority Rule: Each voter votes for one candidate. The candidate with the majority of the votes wins. Majority means MORE than half.

Plurality Method: Each voter votes for one candidate. The candidate with the most votes is the winner.

Condorcet's Method: Each voter ranks the candidates (preference list voting). Each candidate is compared to each of the other candidates. If a candidate wins all of his/her one-on-one contests (is undefeated), he/she is declared the Condorcet winner.

Borda Count: Each voter ranks the n candidates with $n - 1$ points assigned to the first choice, $n - 2$ to the second choice and so on. The candidate with the most points wins. Other rank methods use different point values.

Hare System: If there is no majority winner, then the candidate(s) with the fewest number of first place votes is(are) eliminated and the results are calculated again. If there is still no majority winner, the process continues until a majority winner is found or the remaining candidates are tied.

Sequential Pairwise Voting: Candidates are compared two at a time in a predetermined order known as an agenda. The winner of the pairing is compared to the next candidate on the pre-determined list. This process continues until a winner is determined.

Approval Method: Each voter votes for all the candidates they approve of. The candidate with the most votes wins.

For all methods, a tie-breaking mechanism should be in place prior to the election. Methods could include flipping a coin, drawing straws, number of first place votes, introducing an additional voter, and other methods.

Example

Assume that the following list reflects the voting preferences of all voters.

Pref. List	ADCB	ABCD	BCDA	BCAD	CBDA	CDBA	DCBA
# Voters	3	1	1	1	1	1	1

(a) Who is the majority winner?

9 votes, so majority needs more than $\frac{9}{2} = 4.5$ votes

Candidate	1st place votes
A	$3+1 = 4$
B	$1+1 = 2$
C	$1+1 = 2$
D	$1 = 1$

No majority winner

(b) Who is the plurality winner? A b/c A has the most 1st place votes.

(c) Find the winner using the Hare system.

Pref. List	ADCB	ABCD	~~B~~CDA	~~B~~CAD	CBDA	CDBA	~~D~~CBA
# Voters	3	1	1	1	1	1	1

C wins

Candidate	Votes in 1st round	Votes in 2nd round	Votes in 3rd round
A	4	$4 = 4$	$4 = 4$
B	2	$2 = 2$	Eliminated
C	2	$2+1 = 3$	$3+1+1 = 5$ ✱
D	1 has least, so elim.	Eliminated	—

No majority yet

Pref. List	ADCB	ABCD	BCDA	BCAD	CBDA	CDBA	DCBA
# Voters	3	1	1	1	1	1	1

(d) Who is the Condorcet winner? C C beat all competitors

Choices	Votes	Choices	Votes	Winner
A over B	3+1 =4	B over A	1+1+1+1+1 =5	B
A over C	3+1 =4	C over A	1+1+1+1+1 =5	C
A over D	3+1+1 =5	D over A	1+1+1+1 =4	A
B over C	1+1+1 =3	C over B	3+1+1+1 =6	C
B over D	1+1+1+1 =4	D over B	3+1+1 =5	D
C over D	1+1+1+1+1 =5	D over C	3+1 =4	C

(e) Who is the sequential pairwise winner with the agenda ABCD?

C

B C
C D
C

Pref. List	ADCB	ABCD	BCDA	BCAD	CBDA	CDBA	DCBA
# Voters	3	1	1	1	1	1	1

(f) Who is the Borda count winner? C

	1st place * 3 pts	2nd place * 2 pt	3rd place * 1 pts	Total
A	(3+1)3 = 12		(1)1 = 1	13
B	(1+1)3 = 6	(1+1)2 = 4	(1+1)1 = 2	12
C	(1+1)3 = 6	(1+1+1)2 = 6	(3+1)1 = 4	16
D	(1)3 = 3	(3+1)2 = 8	(1+1)1 = 2	13

Example

Use the chart below to determine what kind of game will be played if each player marks all the games he approves of and the approval method is used to determine the winner.

Game played is Dominos

	A	B	C	D	E	F	G	H	
Pictionary	x		x	x					3
Scrabble	x			x		x		x	4
Dominos		x		x	x	x	x		5 ☆
Trivial Pursuit	x			x	x		x		4
Twister			x						1

Example

A class of 45 students wanted to elect two people to represent them at a meeting. They decided to use the approval method. Use the chart below to determine which two people will be elected. The top row lists the number of voters who approved of the candidate combination in that column

	12	4	9	6	1	7	6
Jeanetta	x			x			x
Mittie		x	x	x			x
Wilton	x		x		x		x
Jamaal	x	x					
Yong		x	x			x	

24

$12+6+6=24$

$4+9+6+6=25$ ☆

$12+9+1+6=28$ ☆

$12+4=16$

$4+9+7=20$

People chosen are Wilton and Mittie

Example

Seventeen board members vote on four candidates, A, B, C, or D, for a new position on their board. Their preference schedules are shown below.

Pref. List	ABCD	DABC	CBDA
# Voters	7	6	4.

17 voters, so we need more than $\frac{17}{2} = 8.5$ for a majority

(a) Who is the majority winner?

Candidate	1st place votes
A	7
B	
C	4
D	6

No majority winner.

(b) Who is the plurality winner?

A b/c A had most votes

(c) Find the winner using the Hare system.

D

B was already eliminated, so went to next choice.

Pref. List	ABCD	DABC	C~~B~~DA
# Voters	7	6	4

Candidate	Votes in 1st round	Votes in 2nd round	Votes in 3rd round
A	7	7	7.
B	0	Eliminated	—
C	4	4	Eliminated b/c Fewest votes
D	6	6	6+4 =10

Pref. List	ABCD	DABC	CBDA

	ABCD	DABC	CBDA
# Voters	7	6	4

(d) Who is the Condorcet winner? No Condorcet winner b/c no one beat all oponents.

Choices	Votes	Choices	Votes	Winner
A over B	7+6 =13	B over A	4 =4	A
A over C	7+6 =13	C over A	4 =4	A
A over D	7 =7	D over A	6+4 =10	D
B over C	7+6 =13	C over B	4 =4	B
B over D	7+4 =11	D over B	6 =6	B
C over D	7+4 =11	D over C	6 =6	C

(e) Who is the sequential pairwise winner with the agenda ADBC?

B

D B

B C

B

Pref. List	ABCD	DABC	CBDA
# Voters	7	6	4

(f) Who is the Borda count winner?

	1st place * 3 pts	2nd place * 2 pt	3rd place * 1 pts	Total
A	(7)3 = 21	(6) 2 = 12		33
B		(7+4)2 = 22	(6)1 = 6	28
C	(4)3 = 12		(7)1 = 7	19
D	(6)3 = 18		(4)1 = 4	22

Sample exam questions would resemble the problems we worked, but may not include the charts.

Exam 4 Review

Example

Ted and Riva must split 8 items between the two of them. They decided to use the adjusted winner procedure. How should these items be divided?

Step 1

Item	Ted	Riva
Movies	20	3
CDs	30	20
Couch	3	~~4~~
Toaster	10	16
TV	15	22
Printer	13	23
Chair	7 Share	10
Rug	2	2

Step 2

Ted: 50 + 2 = 52, $\cancel{+3}$ 55

Riva: 75, $\cancel{-4}$ 71

Step 3

The initial winner is

Riva

The initial loser is

Ted

$\boxed{\text{Riva gets Toaster, TV, Printer + 5.8824\% chair; Ted gets Movies, CD's, Couch, Rug, and 94.1176\% chair}}$

Initially give tied item to person w/ fewer points

Let x = portion of chair that Riva gives to Ted.

Ted = Riva

$55 + 7x = 71 - 10x$

$+10x \qquad +10x$

$55 + 17x = 71$

$-55 \qquad -55$

$17x = 16$

$x = \frac{16}{17} \approx .941176$ or 94.1176%

Riva keeps $1 - \frac{16}{17} = \frac{1}{17}$ of chair

$\approx .058824$ or 5.8824%

Step 4 only list items from initial winner

Item	Point Ratio
Give ★ Couch	$4/3 = 1.\overline{3}$
Toaster	$16/10 = 1.6$
TV	$22/15 = 1.4\overline{6}$
Printer	$23/13 = 1.77$
Share ★ Chair	$10/7 = 1.4$

$\frac{\text{winner's}}{\text{loser's}}$

Give couch b/c it does not change who has more pts.

Lowest pt ratio

Example

Aiden, Beverley, Charlie, and Danielle have inherited a house and a car to share equally. They each submit sealed bids for both items. Describe a fair division of these items using the Knaster Inheritance procedure (tell who gets each item and how much money each person gets or pays).

	House	Car
Step 1	Aiden bid \$120,000 Beverley bid \$140,000 Charlie bid \$150,000 Danielle bid \$115,000	Aiden bid \$8,000 Beverley bid \$7,000 Charlie bid \$6,500 Danielle bid \$8,500
Step 2	Charlie gets the house and places $\frac{3}{4}(150{,}000) = \$112{,}500$ in a holding account.	Danielle gets the car and places $\frac{3}{4}(8{,}500) = \$6375$ in a holding account.
Holding Acct:	\$112,500 − 30000 − 35,000 − 28,750 = 18,750 to split 4 ways	\$6375 − \$2000 − \$1750 − \$1625 = \$1000 to split 4 ways
Steps 3-4	Aiden $\frac{1}{4}(120{,}000) = \$30{,}000$ 4687.50	Aiden $\frac{1}{4}(8{,}000) = \$2000$ 250
	Beverley $\frac{1}{4}(140{,}000) = \$35{,}000$ 4687.50	Beverley $\frac{1}{4}(7{,}000) = \$1750$ 250
	Charlie House − \$112,500 4687.50	Charlie $\frac{1}{4}(6500) = \$1625$ 250
	Danielle $\frac{1}{4}(115{,}000) = \$28{,}750$ 4687.50	Danielle Car − \$6375 250

Step 5

Aiden \$30,000 + \$4687.50 + \$2000 + \$250 = \$36,937.50

Beverley \$35,000 + \$4687.50 + \$1750 + \$250 = \$41,687.50

Charlie House − \$112,500 + \$4687.50 + \$1625 + \$250 = House − \$105,937.50

Danielle \$28,750 + \$4687.50 + Car − \$6375 + \$250 = Car + \$27,312.50

Check yourself: Amt Charlie paid should equal amt other siblings gain

36,937.50 + 41,687.50 + 27,312.50 = 105,937.50

Example Vickrey Auction

Four people were bidding for tickets to a concert. in a Vickrey Auction. Owen bid \$400, Madeline bid \$350, Sofia bid \$420, and Samuel bid \$380.

(a) Who wins the tickets? Sofia (highest bidder)

(b) How much does he/she pay for the tickets?

\$400 (2nd highest bid)

Example

People are bidding on a vacation package on eBay. The minimum bid was set at \$500, and the bid increment is \$8. Complete the following chart to show the progress of the auction before time ran out.

(a)

Bidder	Bid	Current Winner	Current eBay bid
Lily	\$800	Lily	\$500
Nora	\$600	Lily	600+8 = \$608
Devin	\$650	Lily	650+8 = \$658
Nora	\$750	Lily	750+8 = \$758
Samuel	\$850	Samuel	800+8 = \$808
Lily	\$1500	Lily	850+8 = \$858
Samuel	\$950	Lily	950+8 = \$958

2nd highest bid + bid inc.

(b) Who won the auction? Lily

(c) How much did he/she pay for the vacation package? \$958

Example

A county has 11 representatives to apportion to the towns listed below.

(a) Apportion the representatives using the Hamilton method.

$s = \frac{5390}{11} = 490$

Rounded quota $\lfloor q \rfloor$

largest Frac portion of q

Town	Pop.	q			Ham. App
A	1500	$1500/490 = 3.061$	3		3
B	2200	$2200/490 = 4.\boxed{490}$	4	+1	5
C	1640	$1640/490 = 3.347$	3		3
D	50	$50/490 = 0.102$	0		0

total 5390 10 11

11−10=1
rep left to apportion

(b) Apportion the representatives using the Jefferson method.

S & q are same for all methods.

N $\lfloor q \rfloor$

$d_i = \frac{pop}{N+1}$

increase, so largest d_i gets extra seat

Town	Pop.	q	rounded q			Jeff App
A	1500	3.061	3	$1500/(3+1) = 375$		3
B	2200	4.490	4	$2200/(4+1) = 440$	+1	5
C	1640	3.347	3	$1640/(3+1) = 410$		3
D	50	0.102	0	$50/(0+1) = 50$		0

10 11

11−10=1 seat left to apportion

adjusted divisor

d = 440

(c) Apportion the representatives using the Webster method.

S & q are same for all methods

increase, so largest d_i gets extra seat

$N = [q]$ b/c inc → $d_i^+ = \frac{pop}{N+0.5}$

Town	Pop.	q	Rounded q		Web. App
A	1500	3.061	3	$1500/(3+.5) = 428.57$	3
B	2200	4.490	4	$2200/(4+.5) = 488.89$ +1	5
C	1640	3.347	3	$1640/(3+.5) = 468.57$	3
D	50	0.102	0	$50/(0+.5) = 100$	0
			10		11

11−10=1 seat left to apportion

adjusted divisor

$d =$ 488.89

(d) Apportion the representatives using the Hill-Huntington method.

S & q are same for all methods

$q^* = \sqrt{\lfloor q \rfloor \cdot \lceil q \rceil}$

if $q > q^*$, round up ↓ N

dec, so smallest d_i loses a seat

dec → $d_i^- = \frac{pop}{\sqrt{N(N-1)}}$

Town	Pop.	q	q^*	Rounded q		HH App
A	1500	3.061	$\sqrt{3 \cdot 4} = 3.4641$	3	$1500/\sqrt{3(3-1)} = 612.372$	3
B	2200	4.490	$\sqrt{4 \cdot 5} = 4.4721$	5	$2200/\sqrt{5(5-1)} = 491.935$ −1	4
C	1640	3.347	$\sqrt{3 \cdot 4} = 3.4641$	3	$1640/\sqrt{3(3-1)} = 669.527$	3
D	50	0.102	$\sqrt{0 \cdot 1} = 0$	1	$50/\sqrt{1(1-1)}$ = undefined	1
				12		11

11−12 = −1 rep to apportion

$d =$ 491.935

The Jefferson method favors large states. The Hill-Huntington method favors small states.

Example

Label each situation with one of the following five choices:

A. The Alabama paradox occurred.
B. The population paradox occurred.
C. The new states paradox occurred.
D. The quota condition was violated.
E. The quota condition was NOT violated, and no paradox occurred.

(a) A new state was added (along with a proportionate number of representatives) and yielded the following apportionments using the Hamilton method.

E

State	Original Apportionment	New Apportionment
G	8	8
H	5	5
I	3	3
J		2

b/c Hamilton method, we know, it is NOT D

No changes to prev apportionment.

(b) The seats were apportioned using the Jefferson method.

E

State	⌊q⌋	quota	⌈q⌉	Jeff. App.
G	124	124.05	125	124
H	43	43.27	44	43
I	5	5.94	6	6

Not a paradox b/c divisor method

quota condition violated if App is NOT ⌊q⌋ or ⌈q⌉

(c) A new state was added (along with a proportionate number of representatives) and yielded the following apportionments using the Hamilton method.

C

State	Original Apportionment	New Apportionment	
G	8	7	−1
H	5	6	+1
I	3	3	
J		2	

changed prev app

(d) The house size changed from 8 to 9 and yielded the following apportionments using the Hamilton method. not $\emptyset$

A

State	House Size 8	House Size 9
G	5	4 −1
H	3	4 +1
I	0	1 +1

lost a seat when house size inc.

(e) As the population changes, the representation is reapportioned using the Hamilton method. not quota

E

State	Original Apportionment	New Apportionment	Absolute Pop Change	Relative Pop Change
G	14	14	1000	1.1%
H	13	14 +1	1200	3%
I	16	15 −1	1400	2%

State that lost seat did not have higher Relative pop. change than a state that gained.

(f) As the population changes, the representation is reapportioned using the Hamilton method.

B

State	Original Apportionment	New Apportionment	Absolute Pop Change	Relative Pop Change
G	14	14	1000	1.1%
H	13	14 +1	1200	1.5%
I	16	15 −1	1400	2%

State that lost seat had higher Relative pop. change than state that gained.

(g) The seats were apportioned using the Jefferson method.

D

State	⌊q⌋	quota	⌈q⌉	Jeff. App.
G	124	124.95	125	126
H	43	43.27	44	43
I	5	5.34	6	5

no paradox

G's apportionment is $> \lceil q \rceil$