..APTER 13 – FAIR DIVISION

We have three goals for "fairness".

- A fair division procedure is ***equitable*** if each player believes he or she received the same fractional part of the total value.
- A fair division procedure is ***envy-free*** if each player has a strategy that can guarantee him or her a share of whatever is being divided that is, in the eyes of that player, at least as large as that received by any other player, no matter what the other players do.
- A fair division procedure is said to be ***Pareto-optimal*** if it produces an allocation of the property that no other allocation can make one player better off without making some other player worse off.

Adjusted Winner Procedure:

The adjusted winner procedure can be used to divide items between two parties. It achieves all three goals for fairness.

Step 1 Each party distributes 100 points over the items in a way that reflects their relative worth to that party.

Step 2 Each item is initially given to the party that assigns it more points. Each party totals up the number of points it has received. If there is a tie, the item goes to the party with fewer points.

Step 3 If the number of points for each party is equal, the procedure is complete. Otherwise, the party with more points is called the "initial winner" and the other party is called the "initial loser".

Step 4 Calculate the ***point ratio*** for each item that belongs to the initial winner. The point ratio is $\frac{\text{the initial winner's point value for the item}}{\text{the initial loser's point value for the item}}$.

Step 5 Move items from the initial winner to the initial loser in increasing order of point ratio. Stop when you get to an item whose move will cause the initial winner to have fewer points than the initial loser. This item will need to be shared.

Step 6 Let x represent the fractional part of the shared item that will be transferred from the initial winner to the initial loser. Set the initial winner's total points after the sharing of the item equal to the initial loser's total points after the sharing of the item.

...p 7 Solve the equation and state the final division of items between the two parties. Notice that the parties now have an equal number of points of value.

Example

Rand and Mat will split 4 items using the adjusted winner procedure with the point values listed below. How are the items distributed?

Step 1 **Step 2**

Item	Rand	Mat	
Gold coin	25	5	
Saddle Bag	25	20	
Cape	25	35 share	
Hat	25	40	
	50	75	

Step 3 The initial winner is Mat

The initial loser is Rand

Step 4

Item	Point Ratio
Cape	$35/25 = 1.4$ ← smaller
Hat	$40/25 = 1.6$

Steps 5, 6, 7

Let X = the portion of the cape that Mat transfers to Rand.

Mat Rand

$$75 - 35X = 50 + 25X$$
$$+35X \qquad +35X$$
$$75 = 50 + 60X$$
$$-50 \qquad -50$$
$$\frac{25}{60} = \frac{60X}{60}$$
$$\frac{5}{12} = \frac{25}{60} = X$$

So Mat keeps $1 - \frac{5}{12} = \frac{7}{12}$ of the cape

Rand: Gold Coin, Saddle Bag, $\frac{5}{12}$ Cape

Mat: Hat, $\frac{7}{12}$ Cape

Example

Katniss and Peeta will split 5 items using the adjusted winner procedure using the point values listed below. How are the items distributed?

Step 1 **Step 2**

Item	Katniss	Peeta	
Bow and Arrows	50	0	Katniss has $60+20=80$
Water bottle	20 ☆	20 share	
Knife	15	35	Peeta has 75
Food	5	40	
Blanket	10	5	

Step 3 The initial winner is Katniss

The initial loser is Peeta

Step 4

Item	
Bow + Arrows	$5^0/_0 =$ undefined
Water Bottle	$20/20 = 1$ ☆
Blanket	$10/5 = 2$

Katniss
- Bow + Arrows
- Blanket
- $\frac{7}{8}$ water bottle

Peeta
- Knife
- Food
- $\frac{1}{8}$ water Bottle

Steps 5, 6, 7

Let $x =$ portion of water bottle transferred from Katniss to Peeta

$$\begin{aligned}
\text{Katniss} &= \text{Peeta} \\
80 - 20x &= 75 + 20x \\
+20x & \quad\quad +20x \\
80 &= 75 + 40x \\
-75 & \quad -75 \\
\frac{5}{40} &= \frac{40x}{40} \\
\frac{1}{8} = \frac{5}{40} &= x
\end{aligned}$$

So Katniss keeps $1 - \frac{1}{8} = \frac{7}{8}$ of the water bottle

Example

Ozma and Dorothy will split some jewelry using the adjusted winner procedure using the point values listed below. How are the items distributed?

	Step 1		Step 2
Item	Ozma	Dorothy	
Gold Crown	10	5	Oz has 44+11=55
Silver Crown	10	20	
Diamond Bracelet	15	20 share	Dorothy has 84−14=70
Sapphire Bracelet	11	~~14~~ Transfer	
Emerald Bracelet	20	30	
Ruby Bracelet	22	8	
Gold Earrings	12	3	

Step 3 The initial winner is Dorothy

The initial loser is Ozma

Step 4

Item	
Silver Crown	$20/10 = 2$
Diamond Bracelet	$20/15 = 1.\overline{3}$ * share
Sapphire Bracelet	$14/11 = 1.\overline{27}$ * transfer completely
Emerald Bracelet	$30/20 = 1.5$

Steps 5, 6, 7

Let x = portion of DB transferred from D to Oz (Dorothy) (Oz)

$$55 + 15x = 70 - 20x$$
$$+20x \qquad +20x$$
$$55 + 35x = 70$$
$$-55 \qquad -55$$
$$\frac{35x}{35} = \frac{15}{35}$$
$$x = \frac{15}{35} = \frac{3}{7}$$

So Dorothy keeps $1 - \frac{3}{7} = \frac{4}{7}$ of the DB.

Dorothy
- Silver Crown
- Emerald Bracelet
- $\frac{4}{7}$ Diamond Bracelet

Ozma
- Gold Crown
- Sapphire Bracelet
- Ruby Bracelet
- Gold Earrings
- $3/7$ Diamond Bracelet

The Knaster Inheritance Procedure

The Knaster inheritance procedure can be used to divide items among more than two parties. This procedure allocates the items one at a time but requires the parties to have a large amount of cash available.

Step 1 The n heirs – independently and simultaneously – submit monetary bids for the item.

Step 2 The high bidder is awarded the item and places $\left(\frac{n-1}{n}\right)$ (bid) in a holding account.

Step 3 Each of the other heirs withdraws $\frac{1}{n}$ of *his or her own bid* from the holding account.

Step 4 The money left in the holding account is divided equally among all n heirs.

Step 5 The final division of items and cash for the heirs is stated.

<u>Example</u>

Janice, Cindy and Teri receive a coat. To decide who gets the coat they use the Knaster Inheritance Procedure. Janice bids \$90, Cindy bids \$75 and Teri bids \$60. What are the results of the division?

Step 2 ___Janice___ gets the coat and places ___$\frac{2}{3}(90) = \$60$___ in a holding account.

Holding acct: \$60 − 25 − 20 = \$15 to be split Equally 3 ways

Steps 3-4 Janice Coat − \$60 5

Cindy $\frac{1}{3}(75) = 25$ 5

Teri $1/3(60) = 20$ 5

Step 5 Janice Coat − \$55

Cindy \$30

Teri \$25

Example

John, Paul, George, and Ringo receive a piano and a drum set. To decide who gets these items they use the Knaster Inheritance Procedure. What are the results of the division?

	Piano	Drums
Step 1	John bid \$800, Paul bid \$720, George bid \$600, and Ringo bid \$400.	John bid \$500, Paul bid \$440, George bid \$620, and Ringo bid \$400.

Step 2 John gets the piano and places $\frac{3}{4}(800) = \$600$ in a holding account.

George gets the drums and places $\frac{3}{4}(620) = \$465$ in a holding account.

Holding Acct $600 - 180 - 150 - 100 = 170$ to split 4 ways

$465 - 125 - 110 - 100 = \$130$ to split 4 ways

Steps 3-4

	Piano		Drums	
John	Piano − \$600	42.50	$\frac{1}{4}(500) = 125$	32.50
Paul	$\frac{1}{4}(720) = 180$	42.50	$\frac{1}{4}(440) = 110$	32.50
George	$\frac{1}{4}(600) = 150$	42.50	Drums − 465	32.50
Ringo	$\frac{1}{4}(400) = 100$	42.50	$\frac{1}{4}(400) = 100$	32.50

Step 5

John Piano − 600 + 42.50 + 125 + 32.50 = Piano − \$400

Paul \$365

George Drums − \$240

Ringo \$275

Divide and Choose Procedures

With two "players", one player divides the object into two parts then the second player chooses the part he or she wants.

With more players, we can use the Steinhaus Proportional Procedure. For three players, it looks like this.

Step 1 The players (A, B, and C) let player A be the divider.
Step 2 Player A divides the cake into three equal pieces: i, ii, and iii
Step 3 If players B and C each like different pieces, they get those pieces and A gets the remaining piece.
Step 4 If players B and C both want the same piece, they give the least desirable piece to player A. The remaining two pieces are combined. Player B divides the combined pieces and C chooses.

Vickrey Auctions

In a Vickrey auction, bidders independently submit sealed bids for the object being sold. The winner is the high bidder, but he or she pays only the amount of the second-highest bid. For our examples, we will assume that ties do not occur.

Example

Four people were bidding on a new phone. Sally bid \$280, Charles bid \$300, Jace bid \$400, and Beverly bid \$335.

(a) Who wins the phone? Jace

(b) How much does he/she pay for the phone? \$335 eBay uses a variation on Vickrey Auctions for online bidding. An eBay auction has a minimum bid and a bid increment set by the seller before bidding starts. A bidder is free to enter the highest price that he/she is willing to pay for the item, because he/she will only have to pay the amount of the second-highest bid plus the bid increment if he/she wins. Each time a higher bid is placed, the "current eBay bid" is updated to be the second-highest bid plus one bid increment. Bidding continues until time expires.

Example

Susan, Harvey, and Gus are bidding on a copy of the 1903 Longhorn (TAMU's yearbooks were called the Longhorn from 1903 to 1948) on eBay. The minimum bid was set at $60, and the bid increment is $3. Complete the following chart to show the progress of the auction before time ran out.

(a)

Bidder	Bid	Current Winner	Current eBay bid
Harvey	$60	Harvey	$60
Susan	$70	Susan	60+3= $63
Harvey	$66	Susan	66+3= $69
Harvey	$75	Harvey	70+3= $73
Gus	$200	Gus	75+3= $78
Harvey	$90	Gus	90+3= $93
Harvey	$100	Gus	100+3= $103
Susan	$150	Gus	150+3= $153

bid increment

Second highest bid +

(b) Who won the auction? Gus

(c) How much did he/she pay for the yearbook? $153

SAMPLE EXAM QUESTIONS FROM CHAPTER 13

1. Five people were bidding on a piece of land. in a Vickrey Auction Larry bid \$30,000, Terry bid \$25,000, Dane bid \$40,000, Kelly bid \$35,000, and Roland bid \$60,000

(a) Who wins the land? Roland

(b) How much does he/she pay for the land? \$40,000

2. Lucy and Sandy must make a fair division of a printer, a microwave and a lamp. They place point values on the objects as shown below. Using the adjusted winner procedure, what do Lucy and Sandy receive?

Object	Lucy's points		Sandy's points
Printer	40	Share	30
Microwave	10		50
Lamp	50		20
	90		50

Initial winner Lucy

Initial loser Sandy

Point Ratio

Printer $40/30 = 1.\overline{3}$

Lamp $50/20 = 2.5$

Let X = portion of printer transferred from Lucy to Sandy

Lucy = Sandy

$90 - 40X = 50 + 30X$

$+40X \qquad +40X$

$90 = 50 + 70X$

$-50 \qquad -50$

$\frac{40}{70} = \frac{70X}{70}$

$\frac{4}{7} = X$

so Lucy keeps $1 - \frac{4}{7} = \frac{3}{7}$ printer

Lucy:

Lamp

$\frac{3}{7}$ printer

Sandy:

Microwave

$\frac{4}{7}$ printer

3. Nancy, Elayne, and Teri must make a fair division of a boat left to them by their father using the Knaster inheritance procedure. The values they bid on the boat are Nancy - \$4200, Elayne - \$3600, and Teri - \$3000. What are the results of the division?

Nancy gets boat and places $\frac{2}{3}(4200) = 2800$ in holding acct

Holding acct \$2800 − 1200 − 1000 = 600 left to split 3 ways

Nancy Boat − \$2800 200

Elayne $= \frac{1}{3}(3600) =$ \$1200 200

Teri $= \frac{1}{3}(3000) = 1000$ 200

Nancy : Boat − \$2600

Elayne : 1400

Teri : 1200